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PARTIAL DIFFERENTIAL EQUATIONS

-lecture notes-

Partial Differential Equations

Lecture Notes

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Preface

These lecture notes are intented as a straightforward introduction to partial differential equations which can serve as a textbook for undergraduate and beginning graduate students.

For additional reading we recommend following books: W. I. Smirnov [21], I. G. Petrowski [17], P. R. Garabedian [8], W. A. Strauss [23], F. John [10], L. C. Evans [5] and R. Courant and D. Hilbert[4] and D. Gilbarg and N. S. Trudinger [9]. Some material of these lecture notes was taken from some of these books.

Chapter 1

Introduction

Ordinary and partial differential equations occur in many applications. An ordinary differential equation is a special case of a partial differential equation but the behaviour of solutions is quite different in general. It is much more complicated in the case of partial differential equations caused by the fact that the functions for which we are looking at are functions of more than one independent variable.

Equation

$$F(x, y(x), y'(x), \dots, y^{(n)}) = 0$$

is an *ordinary differential equation* of n-th order for the unknown function $y(x)$, where F is given.

An important problem for ordinary differential equations is the *initial value problem*

$$\begin{aligned} y'(x) &= f(x, y(x)) \\ y(x_0) &= y_0 \ , \end{aligned}$$

where f is a given real function of two variables $x,\ y$ and $x_0,\ y_0$ are given real numbers.

Picard-Lindelöf Theorem. *Suppose*
(i) $f(x,y)$ *is continuous in a rectangle*

$$Q = \{(x,y) \in \mathbb{R}^2 : \ |x - x_0| < a, \ |y - y_0| < b\}.$$

(ii) There is a constant K *such that* $|f(x,y)| \le K$ *for all* $(x,y) \in Q$.
(ii) Lipschitz condition: There is a constant L *such that*

$$|f(x, y_2) - f(x, y_1)| \le L|y_2 - y_1|$$

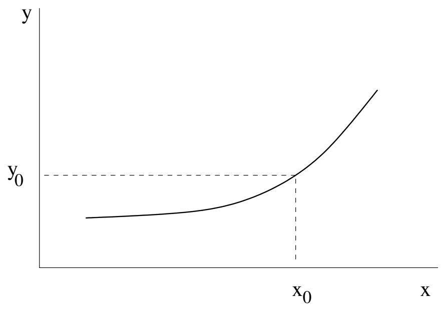


Figure 1.1: Initial value problem

for all $(x, y_1), (x, y_2)$.

Then there exists a unique solution $y \in C^1(x_0-\alpha, x_0+\alpha)$ *of the above initial value problem, where* $\alpha = \min(b/K, a)$.

The linear ordinary differential equation

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots a_1(x)y' + a_0(x)y = 0,$$

where a_j are continuous functions, has exactly n linearly independent solutions. In contrast to this property the partial differential $u_{xx}+u_{yy} = 0$ in $\mathbb{R}^2$ has infinitely many linearly independent solutions in the linear space $C^2(\mathbb{R}^2)$.

The ordinary differential equation of second order

$$y''(x) = f(x, y(x), y'(x))$$

has in general a family of solutions with two free parameters. Thus, it is naturally to consider the associated *initial value problem*

$$\begin{aligned} y''(x) &= f(x, y(x), y'(x)) \\ y(x_0) &= y_0,\ y'(x_0) = y_1, \end{aligned}$$

where y_0 and y_1 are given, or to consider the *boundary value problem*

$$\begin{aligned} y''(x) &= f(x, y(x), y'(x)) \\ y(x_0) &= y_0,\ y(x_1) = y_1. \end{aligned}$$

Initial and boundary value problems play an important role also in the theory of partial differential equations. A *partial differential equation* for

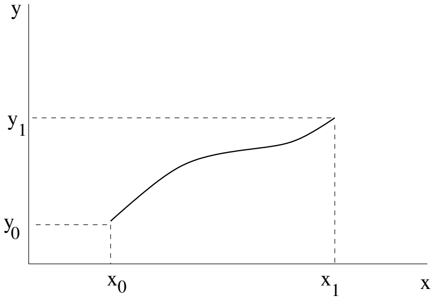


Figure 1.2: Boundary value problem

the unknown function $u(x,y)$ is for example

$$F(x,y,u,u_x,u_y,u_{xx},u_{xy},u_{yy})=0,$$

where the function F is given. This equation is of second order.

An equation is said to be of *n-th order* if the highest derivative which occurs is of order n.

An equation is said to be *linear* if the unknown function and its derivatives are linear in F. For example,

$$a(x,y)u_x+b(x,y)u_y+c(x,y)u=f(x,y),$$

where the functions $a,\ b,\ c$ and f are given, is a linear equation of first order.

An equation is said to be *quasilinear* if it is linear in the highest derivatives. For example,

$$a(x,y,u,u_x,u_y)u_{xx}+b(x,y,u,u_x,u_y)u_{xy}+c(x,y,u,u_x,u_y)u_{yy}=0$$

is a quasilinear equation of second order.

1.1 Examples

1. $u_y=0$, where $u=u(x,y)$. All functions $u=w(x)$ are solutions.

2. $u_x=u_y$, where $u=u(x,y)$. A change of coordinates transforms this equation into an equation of the first example. Set $\xi=x+y,\ \eta=x-y$, then

$$u(x,y)=u\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2}\right)=:v(\xi,\eta).$$

Assume $u \in C^1$, then

$$v_\eta = \frac{1}{2}(u_x - u_y).$$

If $u_x = u_y$, then $v_\eta = 0$ and vice versa, thus $v = w(\xi)$ are solutions for arbitrary C^1-functions $w(\xi)$. Consequently, we have a large class of solutions of the original partial differential equation: $u = w(x + y)$ with an *arbitrary* C^1-function w.

3. A necessary and sufficient condition such that for given C^1-functions $M,\ N$ the integral

$$\int_{P_0}^{P_1} M(x,y)dx + N(x,y)dy$$

is independent of the curve which connects the points P_0 with P_1 in a simply connected domain $\Omega \subset \mathbb{R}^2$ is the partial differential equation (condition of integrability)

$$M_y = N_x$$

in Ω.

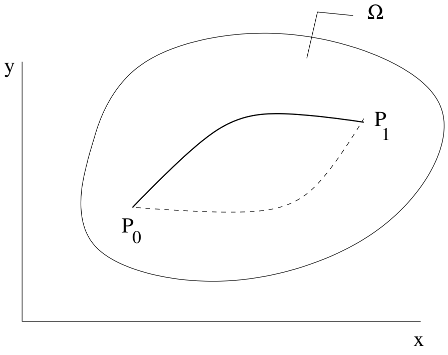


Figure 1.3: Independence of the path

This is one equation for two functions. A large class of solutions is given by $M = \Phi_x,\ N = \Phi_y$, where $\Phi(x,y)$ is an arbitrary C^2-function. It follows from Gauss theorem that these are all C^1-solutions of the above differential equation.

4. *Method of an integrating multiplier for an ordinary differential equation.* Consider the ordinary differential equation

$$M(x,y)dx + N(x,y)dy = 0$$

for given C^1-functions $M,\ N$. Then we seek a C^1-function $\mu(x,y)$ such that $\mu M dx + \mu N dy$ is a total differential, i. e., that $(\mu M)_y = (\mu N)_x$ is satisfied. This is a linear partial differential equation of first order for μ:

$$M\mu_y - N\mu_x = \mu(N_x - M_y).$$

5. Two C^1-functions $u(x,y)$ and $v(x,y)$ are said to be *functionally dependent* if

$$\det\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = 0,$$

which is a linear partial differential equation of first order for u if v is a given C^1-function. A large class of solutions is given by

$$u = H(v(x,y)),$$

where H is an *arbitrary* C^1-function.

6. *Cauchy-Riemann equations.* Set $f(z) = u(x,y)+iv(x,y)$, where $z = x+iy$ and $u,\ v$ are given $C^1(\Omega)$-functions. Here is Ω a domain in $\mathbb{R}^2$. If the function $f(z)$ is differentiable with respect to the complex variable z then $u,\ v$ satisfy the Cauchy-Riemann equations

$$u_x = v_y, \quad u_y = -v_x.$$

It is known from the theory of functions of one complex variable that the real part u and the imaginary part v of a differentiable function $f(z)$ are solutions of the *Laplace equation*

$$\triangle u = 0, \quad \triangle v = 0,$$

where $\triangle u = u_{xx} + u_{yy}$.

7. The *Newton potential*

$$u = \frac{1}{\sqrt{x^2+y^2+z^2}}$$

is a solution of the Laplace equation in $\mathbb{R}^3 \setminus (0,0,0)$, i. e., of

$$u_{xx} + u_{yy} + u_{zz} = 0.$$

8. *Heat equation.* Let $u(x,t)$ be the temperature of a point $x \in \Omega$ at time t, where $\Omega \subset \mathbb{R}^3$ is a domain. Then $u(x,t)$ satisfies in $\Omega \times [0,\infty)$ the *heat equation*

$$u_t = k\triangle u,$$

where $\triangle u = u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3}$ and k is a positive constant. The condition

$$u(x,0) = u_0(x), \quad x \in \Omega,$$

where $u_0(x)$ is given, is an *initial condition* associated to the above heat equation. The condition

$$u(x,t) = h(x,t), \quad x \in \partial\Omega,\ t \geq 0,$$

where $h(x,t)$ is given is a *boundary condition* for the heat equation.

If $h(x,t) = g(x)$, that is, h is independent of t, then one expects that the solution $u(x,t)$ tends to a function $v(x)$ if $t \to \infty$. Moreover, it turns out that v is the solution of the *boundary value problem* for the Laplace equation

$$\begin{aligned} \triangle v &= 0 \ \text{ in } \Omega \\ v &= g(x) \ \text{ on } \partial\Omega. \end{aligned}$$

9. *Wave equation.* The wave equation

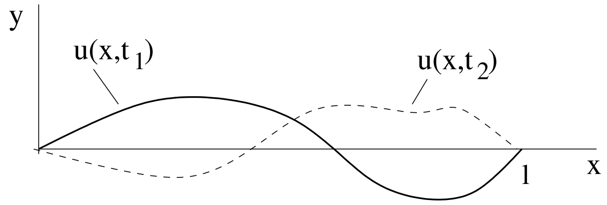


Figure 1.4: Oscillating string

$$u_{tt} = c^2\triangle u,$$

where $u = u(x,t)$, c is a positive constant, describes oscillations of membranes or of three dimensional domains, for example. In the one-dimensional case

$$u_{tt} = c^2 u_{xx}$$

describes oscillations of a string.

Associated *initial conditions* are

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x),$$

where $u_0,\ u_1$ are given functions. Thus the initial position and the initial velocity are prescribed.

If the string is finite one describes additionally *boundary conditions*, for example

$$u(0,t) = 0, \quad u(l,t) = 0 \text{ for all } t \geq 0.$$

1.2 Equations from variational problems

A large class of ordinary and partial differential equations arise from variational problems.

1.2.1 Ordinary differential equations

Set

$$E(v) = \int_a^b f(x, v(x), v'(x))\ dx$$

and for given $u_a,\ u_b \in \mathbb{R}$

$$V = \{v \in C^2[a,b] :\ v(a) = u_a,\ v(b) = u_b\},$$

where $-\infty < a < b < \infty$ and f is sufficiently regular. One of the basic problems in the calculus of variation is

(P) $\qquad \min_{v \in V} E(v).$

Euler equation. *Let $u \in V$ be a solution of (P), then*

$$\frac{d}{dx} f_{u'}(x, u(x), u'(x)) = f_u(x, u(x), u'(x))$$

in (a,b).

Proof. Exercise. Hints: For fixed $\phi \in C^2[a,b]$ with $\phi(a) = \phi(b) = 0$ and real ϵ, $|\epsilon| < \epsilon_0$, set $g(\epsilon) = E(u + \epsilon\phi)$. Since $g(0) \leq g(\epsilon)$ it follows $g'(0) = 0$. Integration by parts in the formula for $g'(0)$ and the following basic lemma in the calculus of variations imply Euler's equation.

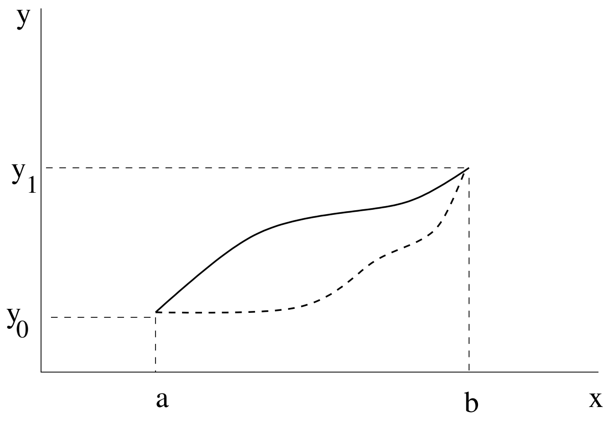


Figure 1.5: Admissible variations

Basic lemma in the calculus of variations. *Let* $h \in C(a,b)$ *and*

$$\int_a^b h(x)\phi(x)\ dx = 0$$

for all $\phi \in C_0^1(a,b)$*. Then* $h(x) \equiv 0$ *on* (a,b)*.*

Proof. Assume $h(x_0) > 0$ for an $x_0 \in (a,b)$, then there is a $\delta > 0$ such that $(x_0-\delta, x_0+\delta) \subset (a,b)$ and $h(x) \geq h(x_0)/2$ on $(x_0-\delta, x_0+\delta)$. Set

$$\phi(x) = \begin{cases} \left(\delta^2 - |x-x_0|^2\right)^2 & \text{if} \quad x \in (x_0-\delta, x_0+\delta) \\ 0 & \text{if} \quad x \in (a,b) \setminus [x_0-\delta, x_0+\delta] \end{cases} .$$

Thus $\phi \in C_0^1(a,b)$ and

$$\int_a^b h(x)\phi(x)\ dx \geq \frac{h(x_0)}{2} \int_{x_0-\delta}^{x_0+\delta} \phi(x)\ dx > 0,$$

which is a contradiction to the assumption of the lemma. □

1.2.2 Partial differential equations

The same procedure as above applied to the following multiple integral leads to a second-order quasilinear partial differential equation. Set

$$E(v) = \int_\Omega F(x, v, \nabla v)\ dx,$$

where $\Omega \subset \mathbb{R}^n$ is a domain, $x = (x_1, \ldots, x_n)$, $v = v(x) : \Omega \mapsto \mathbb{R}$, and $\nabla v = (v_{x_1}, \ldots, v_{x_n})$. Assume that the function F is sufficiently regular in its arguments. For a given function h, defined on $\partial\Omega$, set

$$V = \{v \in C^2(\overline{\Omega}) : \ v = h \text{ on } \partial\Omega\}.$$

Euler equation. *Let $u \in V$ be a solution of (P), then*

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} F_{u_{x_i}} - F_u = 0$$

in Ω.

Proof. Exercise. Hint: Extend the above fundamental lemma of the calculus of variations to the case of multiple integrals. The interval $(x_0 - \delta, x_0 + \delta)$ in the definition of ϕ must be replaced by a ball with center at x_0 and radius δ.

Example: **Dirichlet integral**

In two dimensions the Dirichlet integral is given by

$$D(v) = \int_\Omega \left(v_x^2 + v_y^2\right) \, dxdy$$

and the associated Euler equation is the Laplace equation $\triangle u = 0$ in Ω.

Thus, there is natural relationship between the boundary value problem

$$\triangle u = 0 \ \text{ in } \Omega, \ u = h \ \text{ on } \ \partial\Omega$$

and the variational problem

$$\min_{v \in V} \ D(v).$$

But these problems are not equivalent in general. It can happen that the boundary value problem has a solution but the variational problem has no solution, see for an example Courant and Hilbert [4], Vol. 1, p. 155, where h is a continuous function and the associated solution u of the boundary value problem has no finite Dirichlet integral.

The problems are equivalent, provided the given boundary value function h is in the class $H^{1/2}(\partial\Omega)$, see Lions and Magenes [14].

Example: **Minimal surface equation**

The non-parametric minimal surface problem in two dimensions is to find a minimizer $u = u(x_1, x_2)$ of the problem

$$\min_{v\in V}\int_\Omega \sqrt{1+v_{x_1}^2+v_{x_2}^2}\;dx,$$

where for a given function h defined on the boundary of the domain Ω

$$V=\{v\in C^1(\overline{\Omega}):\ v=h \text{ on } \partial\Omega\}.$$

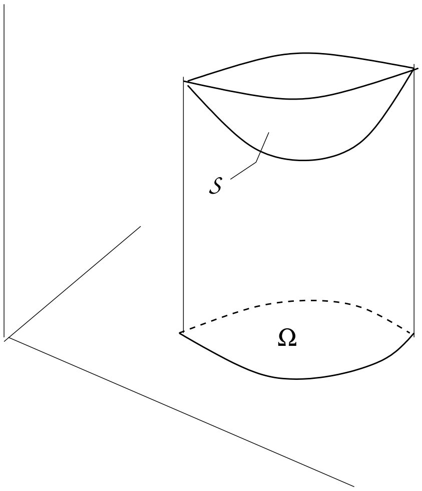


Figure 1.6: Comparison surface

Suppose that the minimizer satisfies the regularity assumption $u \in C^2(\Omega)$, then u is a solution of the *minimal surface equation* (Euler equation) in Ω

$$\frac{\partial}{\partial x_1}\left(\frac{u_{x_1}}{\sqrt{1+|\nabla u|^2}}\right)+\frac{\partial}{\partial x_2}\left(\frac{u_{x_2}}{\sqrt{1+|\nabla u|^2}}\right)=0. \tag{1.1}$$

In fact, the additional assumption $u \in C^2(\Omega)$ is superfluous since it follows from regularity considerations for quasilinear elliptic equations of second order, see for example Gilbarg and Trudinger [9].

Let $\Omega = \mathbb{R}^2$. Each linear function is a solution of the minimal surface equation (1.1). It was shown by Bernstein [2] that there are no other solutions of the minimal surface quation. This is true also for higher dimensions

$n \leq 7$, see Simons [19]. If $n \geq 8$, then there exists also other solutions which define cones, see Bombieri, De Giorgi and Giusti [3].

The linearized minimal surface equation over $u \equiv 0$ is the Laplace equation $\triangle u = 0$. In $\mathbb{R}^2$ linear functions are solutions but also many other functions in contrast to the minimal surface equation. This striking difference is caused by the strong nonlinearity of the minimal surface equation.

More general minimal surfaces are described by using parametric representations. An example is shown in Figure 1.7[1]. See [18], pp. 62, for example, for rotationally symmetric minimal surfaces.

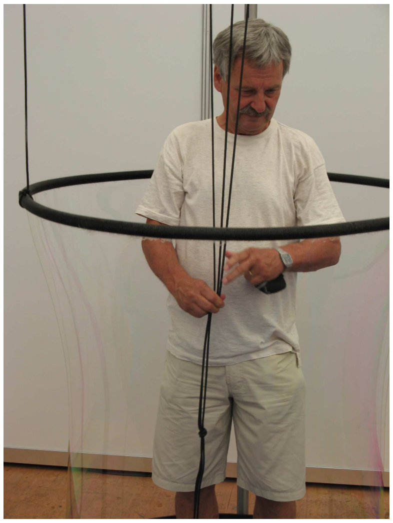


Figure 1.7: Rotationally symmetric minimal surface

Neumann type boundary value problems

Set $V = C^1(\overline{\Omega})$ and

$$E(v) = \int_\Omega F(x, v, \nabla v)\ dx - \int_{\partial\Omega} g(x, v)\ ds,$$

where F and g are given sufficiently regular functions and $\Omega \subset \mathbb{R}^n$ is a bounded and sufficiently regular domain. Assume u is a minimizer of $E(v)$ in V, that is

$$u \in V: \quad E(u) \leq E(v) \ \text{ for all } v \in V,$$

---

[1] An experiment from Beutelspacher's Mathematikum, Wissenschaftsjahr 2008, Leipzig

then

$$\begin{aligned}\int_\Omega \Big(\sum_{i=1}^n F_{u_{x_i}}(x,u,\nabla u)\phi_{x_i} \quad &+ \quad F_u(x,u,\nabla u)\phi\Big)\ dx\\ &- \int_{\partial\Omega} g_u(x,u)\phi\ ds = 0\end{aligned}$$

for all $\phi \in C^1(\overline{\Omega})$. Assume additionally $u \in C^2(\Omega)$, then u is a solution of the Neumann type boundary value problem

$$\begin{aligned}\sum_{i=1}^n \frac{\partial}{\partial x_i} F_{u_{x_i}} - F_u &= 0 \ \text{ in } \Omega\\ \sum_{i=1}^n F_{u_{x_i}}\nu_i - g_u &= 0 \ \text{ on } \partial\Omega,\end{aligned}$$

where $\nu = (\nu_1, \ldots, \nu_n)$ is the exterior unit normal at the boundary $\partial\Omega$. This follows after integration by parts from the basic lemma of the calculus of variations.

Example: **Laplace equation**

Set

$$E(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\ dx - \int_{\partial\Omega} h(x)v\ ds,$$

then the associated boundary value problem is

$$\begin{aligned}\triangle u &= 0 \ \text{ in } \Omega\\ \frac{\partial u}{\partial \nu} &= h \ \text{ on } \partial\Omega.\end{aligned}$$

Example: **Capillary equation**

Let $\Omega \subset \mathbb{R}^2$ and set

$$E(v) = \int_\Omega \sqrt{1 + |\nabla v|^2}\ dx + \frac{\kappa}{2}\int_\Omega v^2\ dx - \cos\gamma \int_{\partial\Omega} v\ ds.$$

Here κ is a positive constant (capillarity constant) and γ is the (constant) boundary contact angle, i. e., the angle between the container wall and the capillary surface, defined by $v = v(x_1, x_2)$, at the boundary. Then the related boundary value problem is

$$\begin{aligned} \mathrm{d}iv\ (Tu) &= \kappa u \ \text{ in } \Omega \\ \nu \cdot Tu &= \cos\gamma \text{ on } \partial\Omega, \end{aligned}$$

where we use the abbreviation

$$Tu = \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}},$$

div (Tu) is the left hand side of the minimal surface equation (1.1) and it is twice the mean curvature of the surface defined by $z = u(x_1, x_2)$, see an exercise.

The above problem describes the ascent of a liquid, water for example, in a vertical cylinder with cross section Ω. Assume the gravity is directed downwards in the direction of the negative x_3-axis. Figure 1.8 shows that liquid can rise along a vertical wedge which is a consequence of the strong nonlinearity of the underlying equations, see Finn [7]. This photo was taken from [15].

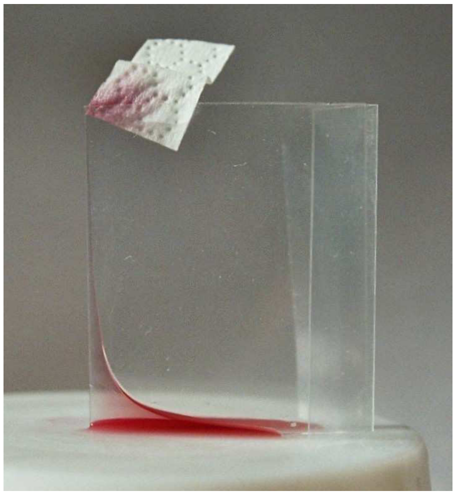


Figure 1.8: Ascent of liquid in a wedge

1.3 Exercises

1. Find nontrivial solutions u of
$$u_x y - u_y x = 0\ .$$

2. Prove: In the linear space $C^2(\mathbb{R}^2)$ there are infinitely many linearly independent solutions of $\triangle u = 0$ in $\mathbb{R}^2$.

   Hint: Real and imaginary part of holomorphic functions are solutions of the Laplace equation.

3. Find all radially symmetric functions which satisfy the Laplace equation in $\mathbb{R}^n \setminus \{0\}$ for $n \geq 2$. A function u is said to be radially symmetric if $u(x) = f(r)$, where $r = (\sum_i^n x_i^2)^{1/2}$.

   Hint: Show that a radially symmetric u satisfies $\triangle u = r^{1-n}\left(r^{n-1}f'\right)'$ by using $\nabla u(x) = f'(r)\frac{x}{r}$.

4. Prove the basic lemma in the calculus of variations: Let $\Omega \subset \mathbb{R}^n$ be a domain and $f \in C(\Omega)$ such that
$$\int_\Omega f(x)h(x)\ dx = 0$$
for all $h \in C_0^2(\Omega)$. Then $f \equiv 0$ in Ω.

5. Write the minimal surface equation (1.1) as a quasilinear equation of second order.

6. Prove that a sufficiently regular minimizer in $C^1(\overline{\Omega})$ of
$$E(v) = \int_\Omega F(x, v, \nabla v)\ dx - \int_{\partial\Omega} g(v, v)\ ds,$$
is a solution of the boundary value problem
$$\begin{aligned} \sum_{i=1}^n \frac{\partial}{\partial x_i} F_{u_{x_i}} - F_u &= 0 \ \text{ in } \Omega \\ \sum_{i=1}^n F_{u_{x_i}} \nu_i - g_u &= 0 \ \text{ on } \partial\Omega, \end{aligned}$$
where $\nu = (\nu_1, \ldots, \nu_n)$ is the exterior unit normal at the boundary $\partial\Omega$.

7. Prove that $\nu \cdot Tu = \cos\gamma$ on $\partial\Omega$, where γ is the angle between the container wall, which is here a cylinder, and the surface S, defined by $z = u(x_1, x_2)$, at the boundary of S, ν is the exterior normal at $\partial\Omega$.

   Hint: The angle between two surfaces is by definition the angle between the two associated normals at the intersection of the surfaces.

8. Let Ω be bounded and assume $u \in C^2(\overline{\Omega})$ is a solution of
$$\begin{aligned} \mathrm{d}iv\ Tu &= C \text{ in } \Omega \\ \nu \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} &= \cos\gamma \text{ on } \partial\Omega, \end{aligned}$$
   where C is a constant.

   Prove that
$$C = \frac{|\partial\Omega|}{|\Omega|}\cos\gamma\ .$$

   Hint: Integrate the differential equation over Ω.

9. Assume $\Omega = B_R(0)$ is a disc with radius R and the center at the origin. Show that radially symmetric solutions $u(x) = w(r)$, $r = \sqrt{x_1^2 + x_2^2}$, of the capillary boundary value problem are solutions of
$$\begin{aligned} \left(\frac{rw'}{\sqrt{1+w'^2}}\right)' &= \kappa r w \quad \text{in } 0 < r < R \\ \frac{w'}{\sqrt{1+w'^2}} &= \cos\gamma \quad \text{if } r = R. \end{aligned}$$

   Remark. It follows from a maximum principle of Concus and Finn [7] that a solution of the capillary equation over a disc must be radially symmetric.

10. Find all radially symmetric solutions of
$$\begin{aligned} \left(\frac{rw'}{\sqrt{1+w'^2}}\right)' &= Cr \quad \text{in } 0 < r < R \\ \frac{w'}{\sqrt{1+w'^2}} &= \cos\gamma \quad \text{if } r = R. \end{aligned}$$

    Hint: From an exercise above it follows that
$$C = \frac{2}{R}\cos\gamma.$$

11. Show that div Tu is twice the mean curvature of the surface defined by $z = u(x_1, x_2)$.

Chapter 2

Equations of first order

For a given sufficiently regular function F the general equation of first order for the unknown function $u(x)$ is

$$F(x, u, \nabla u) = 0$$

in $\Omega \in \mathbb{R}^n$. The main tool for studying related problems is the theory of ordinary differential equations. This is quite different for systems of partial differential of first order.

The general linear partial differential equation of first order can be written as

$$\sum_{i=1}^{n} a_i(x) u_{x_i} + c(x) u = f(x)$$

for given functions a_i, c and f. The general quasilinear partial differential equation of first order is

$$\sum_{i=1}^{n} a_i(x, u) u_{x_i} + c(x, u) = 0.$$

2.1 Linear equations

Let us begin with the linear homogeneous equation

$$a_1(x, y) u_x + a_2(x, y) u_y = 0. \tag{2.1}$$

Assume there is a C^1-solution $z = u(x, y)$. This function defines a surface S which has at $P = (x, y, u(x, y))$ the normal

$$\mathbf{N} = \frac{1}{\sqrt{1 + |\nabla u|^2}}(-u_x, -u_y, 1)$$

and the tangential plane defined by

$$\zeta - z = u_x(x,y)(\xi - x) + u_y(x,y)(\eta - y).$$

Set $p = u_x(x,y)$, $q = u_y(x,y)$ and $z = u(x,y)$. The tuple (x,y,z,p,q) is called *surface element* and the tuple (x,y,z) *support* of the surface element. The tangential plane is defined by the surface element. On the other hand, differential equation (2.1)

$$a_1(x,y)p + a_2(x,y)q = 0$$

defines at each support (x,y,z) a bundle of planes if we consider all (p,q) satisfying this equation. For fixed (x,y), this family of planes $\Pi(\lambda) = \Pi(\lambda; x,y)$ is defined by a one parameter family of ascents $p(\lambda) = p(\lambda; x,y)$, $q(\lambda) = q(\lambda; x,y)$. The envelope of these planes is a line since

$$a_1(x,y)p(\lambda) + a_2(x,y)q(\lambda) = 0,$$

which implies that the normal $\mathbf{N}(\lambda)$ on $\Pi(\lambda)$ is perpendicular on $(a_1, a_2, 0)$.

Consider a curve $\mathbf{x}(\tau) = (x(\tau), y(\tau), z(\tau))$ on $\mathcal{S}$, let $T_{\mathbf{x}_0}$ be the tangential plane at $\mathbf{x}_0 = (x(\tau_0), y(\tau_0), z(\tau_0))$ of $\mathcal{S}$ and consider on $T_{\mathbf{x}_0}$ the line

$$L: \quad l(\sigma) = \mathbf{x}_0 + \sigma \mathbf{x}'(\tau_0), \quad \sigma \in \mathbb{R},$$

see Figure 2.1.

We assume L coincides with the envelope, which is a line here, of the family of planes $\Pi(\lambda)$ at (x,y,z). Assume that $T_{\mathbf{x}_0} = \Pi(\lambda_0)$ and consider two planes

$$\begin{array}{rrcl} \Pi(\lambda_0): & z - z_0 & = & (x - x_0)p(\lambda_0) + (y - y_0)q(\lambda_0) \\ \Pi(\lambda_0 + h): & z - z_0 & = & (x - x_0)p(\lambda_0 + h) + (y - y_0)q(\lambda_0 + h). \end{array}$$

At the intersection $l(\sigma)$ we have

$$(x - x_0)p(\lambda_0) + (y - y_0)q(\lambda_0) = (x - x_0)p(\lambda_0 + h) + (y - y_0)q(\lambda_0 + h).$$

Thus,

$$x'(\tau_0)p'(\lambda_0) + y'(\tau_0)q'(\lambda_0) = 0.$$

From the differential equation

$$a_1(x(\tau_0), y(\tau_0))p(\lambda) + a_2(x(\tau_0), y(\tau_0))q(\lambda) = 0$$

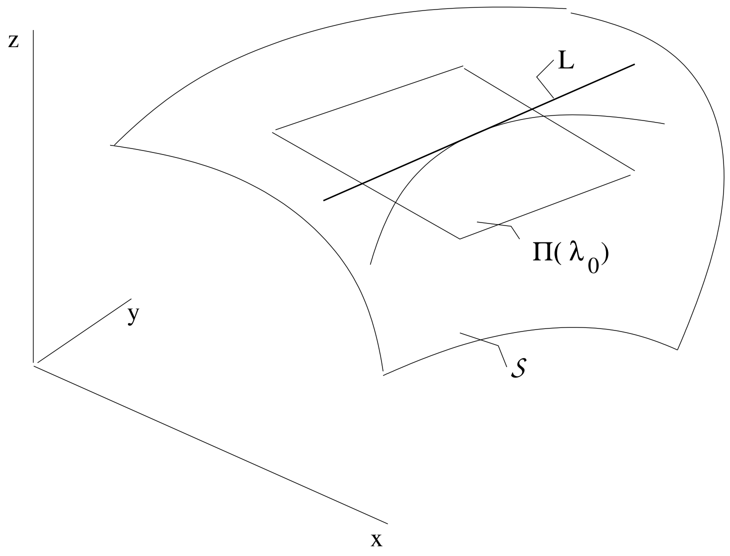


Figure 2.1: Curve on a surface

it follows

$$a_1p'(\lambda_0)+a_2q'(\lambda_0)=0.$$

Consequently

$$(x'(\tau),y'(\tau))=\frac{x'(\tau)}{a_1(x(\tau,y(\tau))}(a_1(x(\tau),y(\tau)),a_2(x(\tau),y(\tau)),$$

since τ_0 was an arbitrary parameter. Here we assume that $x'(\tau)\neq 0$ and $a_1(x(\tau),y(\tau))\neq 0$.

Then we introduce a new parameter t by the inverse of $\tau=\tau(t)$, where

$$t(\tau)=\int_{\tau_0}^{\tau}\frac{x'(s)}{a_1(x(s),y(s))}\ ds.$$

It follows $x'(t)=a_1(x,y),\ y'(t)=a_2(x,y)$. We denote $\mathbf{x}(\tau(t))$ by $\mathbf{x}(t)$ again.

Now we consider the initial value problem

$$x'(t)=a_1(x,y),\quad y'(t)=a_2(x,y),\quad x(0)=x_0,\ y(0)=y_0. \tag{2.2}$$

From the theory of ordinary differential equations it follows (Theorem of Picard-Lindelöf) that there is a unique solution in a neighbouhood of $t=0$ provided the functions $a_1,\ a_2$ are in C^1. From this definition of the curves

$(x(t), y(t))$ is follows that the field of directions $(a_1(x_0, y_0), a_2(x_0, y_0))$ defines the slope of these curves at $(x(0), y(0))$.

Definition. The differential equations in (2.2) are called *characteristic equations* or characteristic system and solutions of the associated initial value problem are called *characteristic curves.*

Definition. A function $\phi(x, y)$ is said to be an *integral* of the characteristic system if $\phi(x(t), y(t)) = const.$ for each characteristic curve. The constant depends on the characteristic curve considered.

Proposition 2.1. *Assume $\phi \in C^1$ is an integral, then $u = \phi(x, y)$ is a solution of (2.1).*

Proof. Consider for given (x_0, y_0) the above initial value problem (2.2). Since $\phi(x(t), y(t)) = const.$ it follows

$$\phi_x x' + \phi_y y' = 0$$

for $|t| < t_0$, $t_0 > 0$ and sufficiently small. Thus

$$\phi_x(x_0, y_0)a_1(x_0, y_0) + \phi_y(x_0, y_0)a_2(x_0, y_0) = 0.$$

□

Remark. If $\phi(x, y)$ is a solution of equation (2.1) then also $H(\phi(x, y))$, where $H(s)$ is a given C^1-function.

Examples

1. Consider

$$a_1 u_x + a_2 u_y = 0,$$

where $a_1,\ a_2$ are constants. The system of characteristic equations is

$$x' = a_1,\ y' = a_2.$$

Thus the characteristic curves are parallel straight lines defined by

$$x = a_1 t + A,\ y = a_2 t + B,$$

where $A,\ B$ are arbitrary constants. From these equations it follows that

$$\phi(x,y) := a_2x - a_1y$$

is constant along each characteristic curve. Consequently, see Proposition 2.1, $u = a_2x - a_1y$ is a solution of the differential equation. From an exercise it follows that

$$u = H(a_2x - a_1y), \tag{2.3}$$

where $H(s)$ is an arbitrary C^1-function, is also a solution. Since u is constant when $a_2x - a_1y$ is constant, equation (2.3) defines cylinder surfaces which are generated by parallel straight lines which are parallel to the (x,y)-plane, see Figure 2.2.

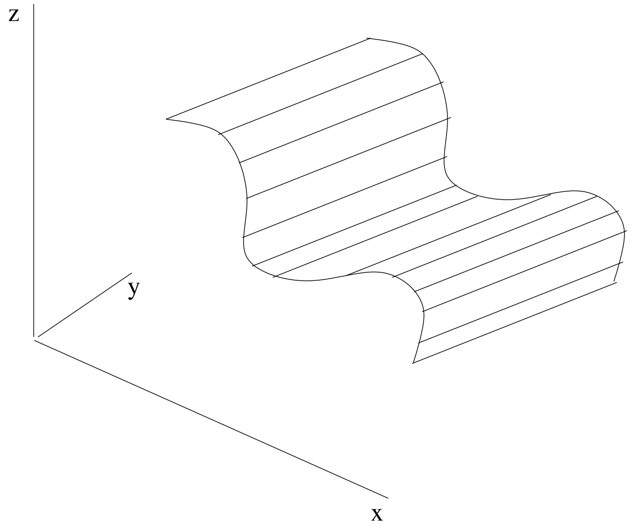


Figure 2.2: Cylinder surfaces

2. Consider the differential equation

$$xu_x + yu_y = 0.$$

The characteristic equations are

$$x' = x,\ y' = y,$$

and the characteristic curves are given by

$$x = Ae^t,\ y = Be^t,$$

where $A,\ B$ are arbitrary constants. Thus, an integral is y/x, $x \neq 0$, and for a given C^1-function the function $u = H(x/y)$ is a solution of the differential equation. If $y/x = const.$, then u is constant. Suppose that $H'(s) > 0$, for example, then u defines right helicoids (in German: Wendelflächen), see Figure 2.3

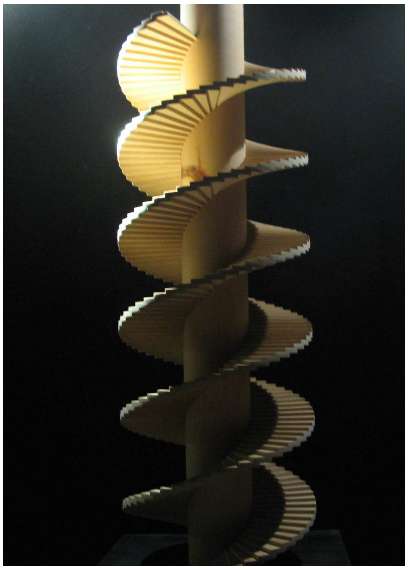


Figure 2.3: Right helicoid, $a^2 < x^2 + y^2 < R^2$ (Museo Ideale Leonardo da Vinci, Italy)

3. Consider the differential equation

$$yu_x - xu_y = 0.$$

The associated characteristic system is

$$x' = y,\ y' = -x.$$

If follows

$$x'x + yy' = 0,$$

or, equivalently,

$$\frac{d}{dt}(x^2+y^2)=0,$$

which implies that $x^2+y^2=const.$ along each characteristic. Thus, rotationally symmetric surfaces defined by $u=H(x^2+y^2)$, where $H'\neq 0$, are solutions of the differential equation.

4. The associated characteristic equations to

$$ayu_x+bxu_y=0,$$

where $a,\ b$ are positive constants, are given by

$$x'=ay,\ y'=bx.$$

It follows $bxx'-ayy'=0$, or equivalently,

$$\frac{d}{dt}(bx^2-ay^2)=0.$$

Solutions of the differential equation are $u=H(bx^2-ay^2)$, which define surfaces which have a hyperbola as the intersection with planes parallel to the (x,y)-plane. Here $H(s)$ is an arbitrary C^1-function, $H'(s)\neq 0$.

2.2 Quasilinear equations

Here we consider the equation

$$a_1(x,y,u)u_x+a_2(x,y,u)u_y=a_3(x,y,u). \tag{2.4}$$

The inhomogeneous linear equation

$$a_1(x,y)u_x+a_2(x,y)u_y=a_3(x,y)$$

is a special case of (2.4).

One arrives at characteristic equations $x'=a_1,\ y'=a_2,\ z'=a_3$ from (2.4) by the same arguments as in the case of homogeneous linear equations in two variables. The additional equation $z'=a_3$ follows from

$$\begin{aligned} z'(\tau) &= p(\lambda)x'(\tau)+q(\lambda)y'(\tau)\\ &= pa_1+qa_2\\ &= a_3, \end{aligned}$$

see also Section 2.3, where the general case of nonlinear equations in two variables is considered.

2.2.1 A linearization method

We can transform the inhomogeneous equation (2.4) into a homogeneous linear equation for an unknown function of three variables by the following trick.

We are looking for a function $\psi(x, y, u)$ such that the solution $u = u(x, y)$ of (2.4) is defined implicitly by $\psi(x, y, u) = const$. Assume there is such a function ψ and let u be a solution of (2.4), then

$$\psi_x + \psi_u u_x = 0, \quad \psi_y + \psi_u u_y = 0.$$

Assume $\psi_u \neq 0$, then

$$u_x = -\frac{\psi_x}{\psi_u}, \quad u_y = -\frac{\psi_y}{\psi_u}.$$

From (2.4) we obtain

$$a_1(x, y, z)\psi_x + a_2(x, y, z)\psi_y + a_3(x, y, z)\psi_z = 0, \tag{2.5}$$

where $z := u$.

We consider the associated system of characteristic equations

$$\begin{aligned} x'(t) &= a_1(x, y, z) \\ y'(t) &= a_2(x, y, z) \\ z'(t) &= a_3(x, y, z). \end{aligned}$$

One arrives at this system by the same arguments as in the two-dimensional case above.

Proposition 2.2. (i) *Assume $w \in C^1$, $w = w(x, y, z)$, is an integral, i. e., it is constant along each fixed solution of (2.5), then $\psi = w(x, y, z)$ is a solution of (2.5).*
(ii) *The function $z = u(x, y)$, implicitly defined through $\psi(x, u, z) = const.$, is a solution of (2.4), provided that $\psi_z \neq 0$.*
(iii) *Let $z = u(x, y)$ be a solution of (2.4) and let $(x(t), y(t))$ be a solution of*

$$x'(t) = a_1(x, y, u(x, y)), \quad y'(t) = a_2(x, y, u(x, y)),$$

then $z(t) := u(x(t), y(t))$ satisfies the third of the above characteristic equations.

Proof. Exercise.

2.2.2 Initial value problem of Cauchy

Consider again the quasilinear equation

$$(\star)\qquad a_1(x,y,u)u_x + a_2(x,y,u)u_y = a_3(x,y,u).$$

Let

$$\Gamma: \quad x = x_0(s),\ y = y_0(s),\ z = z_0(s),\ s_1 \le s \le s_2,\ -\infty < s_1 < s_2 < +\infty$$

be a regular curve in $\mathbb{R}^3$ and denote by $\mathcal{C}$ the orthogonal projection of Γ onto the (x,y)-plane, i. e.,

$$\mathcal{C}: \quad x = x_0(s), \quad y = y_0(s).$$

Initial value problem of Cauchy: *Find a C^1-solution $u = u(x,y)$ of $(\star)$ such that $u(x_0(s), y_0(s)) = z_0(s)$, i. e., we seek a surface $\mathcal{S}$ defined by $z = u(x,y)$ which contains the curve Γ.*

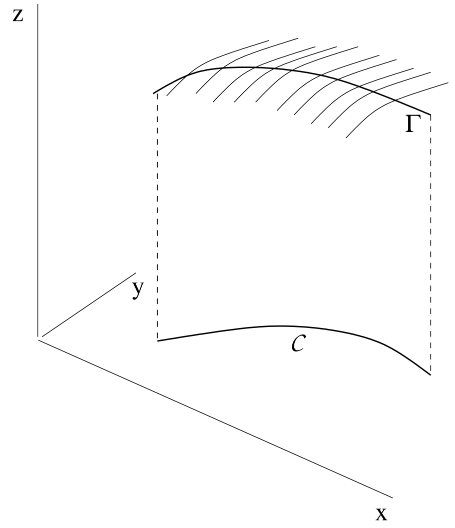


Figure 2.4: Cauchy initial value problem

Definition. The curve Γ is said to be *noncharacteristic* if

$$x_0'(s)a_2(x_0(s), y_0(s)) - y_0'(s)a_1(x_0(s), y_0(s)) \neq 0.$$

Theorem 2.1. *Assume $a_1,\ a_2,\ a_2 \in C^1$ in their arguments, the initial data $x_0,\ y_0,\ z_0 \in C^1[s_1, s_2]$ and Γ is noncharacteristic.*

Then there is a neighbourhood of $\mathcal{C}$ such that there exists exactly one solution u of the Cauchy initial value problem.

Proof. (i) Existence. Consider the following initial value problem for the system of characteristic equations to $(\star)$:

$$\begin{aligned} x'(t) &= a_1(x,y,z) \\ y'(t) &= a_2(x,y,z) \\ z'(t) &= a_3(x,y,z) \end{aligned}$$

with the initial conditions

$$\begin{aligned} x(s,0) &= x_0(s) \\ y(s,0) &= y_0(s) \\ z(s,0) &= z_0(s). \end{aligned}$$

Let $x = x(s,t)$, $y = y(s,t)$, $z = z(s,t)$ be the solution, $s_1 \le s \le s_2$, $|t| < \eta$ for an $\eta > 0$. We will show that this set of strings sticked onto the curve Γ, see Figure 2.4, defines a surface. To show this, we consider the inverse functions $s = s(x,y)$, $t = t(x,y)$ of $x = x(s,t)$, $y = y(s,t)$ and show that $z(s(x,y),t(x,y))$ is a solution of the initial problem of Cauchy. The inverse functions s and t exist in a neighbourhood of $t = 0$ since

$$\det \frac{\partial(x,y)}{\partial(s,t)}\Big|_{t=0} = \begin{vmatrix} x_s & x_t \\ y_s & y_t \end{vmatrix}_{t=0} = x_0'(s)a_2 - y_0'(s)a_1 \neq 0,$$

and the initial curve Γ is noncharacteristic by assumption.

Set

$$u(x,y) := z(s(x,y),t(x,y)),$$

then u satisfies the initial condition since

$$u(x,y)|_{t=0} = z(s,0) = z_0(s).$$

The following calculation shows that u is also a solution of the differential equation $(\star)$.

$$\begin{aligned} a_1u_x + a_2u_y &= a_1(z_ss_x + z_tt_x) + a_2(z_ss_y + z_tt_y) \\ &= z_s(a_1s_x + a_2s_y) + z_t(a_1t_x + a_2t_y) \\ &= z_s(s_xx_t + s_yy_t) + z_t(t_xx_t + t_yy_t) \\ &= a_3 \end{aligned}$$

since $0 = s_t = s_x x_t + s_y y_t$ and $1 = t_t = t_x x_t + t_y y_t$.

(ii) Uniqueness. Suppose that $v(x, y)$ is a second solution. Consider a point (x', y') in a neighbourhood of the curve $(x_0(s), y(s))$, $s_1 - \epsilon \le s \le s_2 + \epsilon$, $\epsilon > 0$ small. The inverse parameters are $s' = s(x', y')$, $t' = t(x', y')$, see Figure 2.5.

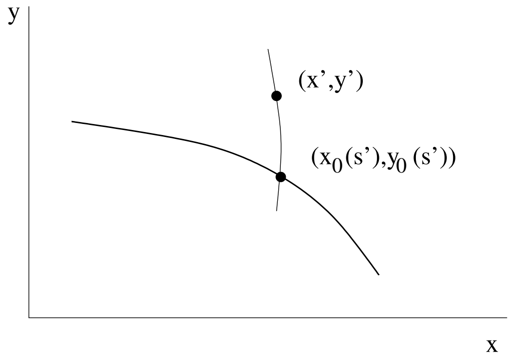


Figure 2.5: Uniqueness proof

Let

$$\mathcal{A}: \quad x(t) := x(s', t),\ y(t) := y(s', t),\ z(t) := z(s', t)$$

be the solution of the above initial value problem for the characteristic differential equations with the initial data

$$x(s', 0) = x_0(s'),\ y(s', 0) = y_0(s'),\ z(s', 0) = z_0(s').$$

According to its construction this curve is on the surface $\mathcal{S}$ defined by $u = u(x, y)$ and $u(x', y') = z(s', t')$. Set

$$\psi(t) := v(x(t), y(t)) - z(t),$$

then

$$\begin{aligned} \psi'(t) &= v_x x' + v_y y' - z' \\ &= x_x a_1 + v_y a_2 - a_3 = 0 \end{aligned}$$

and

$$\psi(0) = v(x(s', 0), y(s', 0)) - z(s', 0) = 0$$

since v is a solution of the differential equation and satisfies the initial condition by assumption. Thus, $\psi(t) \equiv 0$, i. e.,

$$v(x(s', t), y(s', t)) - z(s', t) = 0.$$

Set $t = t'$, then

$$v(x', y') - z(s', t') = 0,$$

which shows that $v(x', y') = u(x', y')$ because of $z(s', t') = u(x', y')$. $\Box$

Remark. In general, there is no uniqueness if the initial curve Γ is a characteristic curve, see an exercise and Figure 2.6 which illustrates this case.

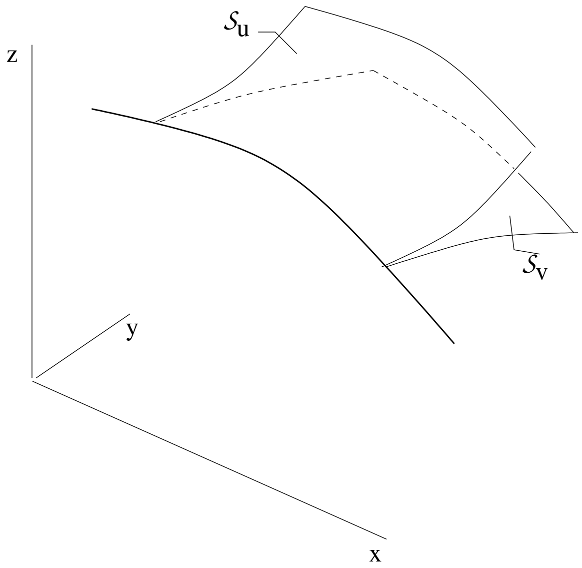


Figure 2.6: Multiple solutions

Examples

1. Consider the Cauchy initial value problem

$$u_x + u_y = 0$$

with the initial data

$$x_0(s) = s,\ y_0(s) = 1,\ z_0(s) \text{ is a given } C^1\text{-function}.$$

These initial data are noncharacteristic since $y_0' a_1 - x_0' a_2 = -1$. The solution of the associated system of characteristic equations

$$x'(t) = 1,\ y'(t) = 1,\ u'(t) = 0$$

with the initial conditions

$$x(s,0)=x_0(s),\ y(s,0)=y_0(s),\ z(s,0)=z_0(s)$$

is given by

$$x=t+x_0(s),\ y=t+y_0(s),\ z=z_0(s),$$

i. e.,

$$x=t+s,\ y=t+1,\ z=z_0(s).$$

It follows $s=x-y+1,\ t=y-1$ and that $u=z_0(x-y+1)$ is the solution of the Cauchy initial value problem.

2. A problem from kinetics in chemistry. Consider for $x\ge 0,\ y\ge 0$ the problem

$$u_x+u_y=\left(k_0\mathrm{e}^{-k_1x}+k_2\right)(1-u)$$

with initial data

$$u(x,0)=0,\ x>0,\ \text{and } u(0,y)=u_0(y),\ y>0.$$

Here the constants k_j are positive, these constants define the velocity of the reactions in consideration, and the function $u_0(y)$ is given. The variable x is the time and y is the hight of a tube, for example, in which the chemical reaction takes place, and u is the concentration of the chemical substance.

In contrast to our previous assumptions, the initial data are not in C^1. The projection $\mathcal{C}_1\cup\mathcal{C}_2$ of the initial curve onto the (x,y)-plane has a corner at the origin, see Figure 2.7.

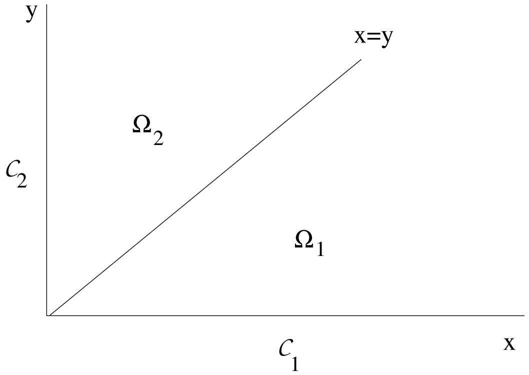


Figure 2.7: Domains to the chemical kinetics example

The associated system of characteristic equations is

$$x'(t) = 1,\ y'(t) = 1,\ z'(t) = \left(k_0 \mathrm{e}^{-k_1 x} + k_2\right)(1 - z).$$

It follows $x = t + c_1$, $y = t + c_2$ with constants c_j. Thus the projection of the characteristic curves on the (x, y)-plane are straight lines parallel to $y = x$. We will solve the initial value problems in the domains Ω_1 and Ω_2, see Figure 2.7, separately.

(i) *The initial value problem in* Ω_1. The initial data are

$$x_0(s) = s,\ y_0(s) = 0,\ z_0(0) = 0,\ s \geq 0.$$

It follows

$$x = x(s, t) = t + s,\ y = y(s, t) = t.$$

Thus

$$z'(t) = (k_0 \mathrm{e}^{-k_1(t+s)} + k_2)(1 - z),\ z(0) = 0.$$

The solution of this initial value problem is given by

$$z(s, t) = 1 - \exp\left(\frac{k_0}{k_1}\mathrm{e}^{-k_1(s+t)} - k_2 t - \frac{k_0}{k_1}\mathrm{e}^{-k_1 s}\right).$$

Consequently

$$u_1(x, y) = 1 - \exp\left(\frac{k_0}{k_1}\mathrm{e}^{-k_1 x} - k_2 y - k_0 k_1 \mathrm{e}^{-k_1(x-y)}\right)$$

is the solution of the Cauchy initial value problem in Ω_1. If time x tends to ∞, we get the limit

$$\lim_{x \to \infty} u_1(x, y) = 1 - \mathrm{e}^{-k_2 y}.$$

(ii) *The initial value problem in* Ω_2. The initial data are here

$$x_0(s) = 0,\ y_0(s) = s,\ z_0(0) = u_0(s),\ s \geq 0.$$

It follows

$$x = x(s, t) = t,\ y = y(s, t) = t + s.$$

Thus

$$z'(t) = (k_0 \mathrm{e}^{-k_1 t} + k_2)(1 - z),\ z(0) = 0.$$

The solution of this initial value problem is given by

$$z(s,t)=1-(1-u_0(s))\exp\left(\frac{k_0}{k_1}\mathrm{e}^{-k_1t}-k_2t-\frac{k_0}{k_1}\right).$$

Consequently

$$u_2(x,y)=1-(1-u_0(y-x))\exp\left(\frac{k_0}{k_1}\mathrm{e}^{-k_1x}-k_2x-\frac{k_0}{k_1}\right)$$

is the solution in Ω_2.

If $x=y$, then

$$\begin{aligned} u_1(x,y) &= 1-\exp\left(\frac{k_0}{k_1}\mathrm{e}^{-k_1x}-k_2x-\frac{k_0}{k_1}\right) \\ u_2(x,y) &= 1-(1-u_0(0))\exp\left(\frac{k_0}{k_1}\mathrm{e}^{-k_1x}-k_2x-\frac{k_0}{k_1}\right). \end{aligned}$$

If $u_0(0)>0$, then $u_1<u_2$ if $x=y$, i. e., there is a jump of the concentration of the substrate along its burning front defined by $x=y$.

Remark. Such a problem with discontinuous initial data is called *Riemann problem.* See an exercise for another Riemann problem.

The case that a solution of the equation is known

Here we will see that we get immediately a solution of the Cauchy initial value problem if a solution of the *homogeneous linear equation*

$$a_1(x,y)u_x+a_2(x,y)u_y=0$$

is known.

Let

$$x_0(s),\ y_0(s),\ z_0(s),\ s_1<s<s_2$$

be the initial data and let $u=\phi(x,y)$ be a solution of the differential equation. We assume that

$$\phi_x(x_0(s),y_0(s))x_0'(s)+\phi_y(x_0(s),y_0(s))y_0'(s)\neq 0$$

is satisfied. Set $g(s)=\phi(x_0(s),y_0(s))$ and let $s=h(g)$ be the inverse function.

The solution of the Cauchy initial problem is given by $u_0\left(h(\phi(x,y))\right)$.

This follows since in the problem considered a composition of a solution is a solution again, see an exercise, and since

$$u_0\left(h(\phi(x_0(s),y_0(s))\right) = u_0(h(g)) = u_0(s).$$

Example: Consider equation

$$u_x + u_y = 0$$

with initial data

$$x_0(s) = s,\ y_0(s) = 1,\ u_0(s) \text{ is a given function.}$$

A solution of the differential equation is $\phi(x,y) = x - y$. Thus

$$\phi((x_0(s),y_0(s)) = s - 1$$

and

$$u_0(\phi + 1) = u_0(x - y + 1)$$

is the solution of the problem.

2.3 Nonlinear equations in two variables

Here we consider equation

$$F(x,y,z,p,q) = 0, \tag{2.6}$$

where $z = u(x,y)$, $p = u_x(x,y)$, $q = u_y(x,y)$ and $F \in C^2$ is given such that $F_p^2 + F_q^2 \neq 0$.

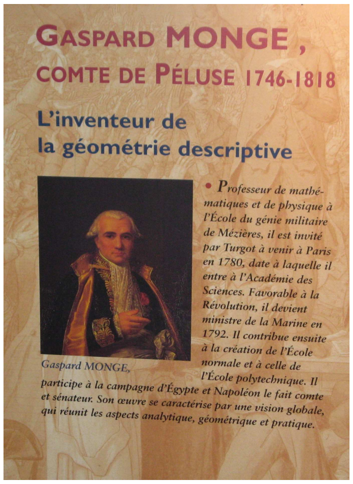
In contrast to the quasilinear case, this general nonlinear equation is more complicated. Together with (2.6) we will consider the following system of ordinary equations which follow from considerations below as necessary conditions, in particular from the assumption that there is a solution of (2.6).

$$\begin{aligned}
x'(t) &= F_p && (2.7)\\
y'(t) &= F_q && (2.8)\\
z'(t) &= pF_p + qF_q && (2.9)\\
p'(t) &= -F_x - F_u p && (2.10)\\
q'(t) &= -F_y - F_u q. && (2.11)
\end{aligned}$$

Definition. Equations (2.7)–(2.11) are said to be *characteristic equations* of equation (2.6) and a solution

$$(x(t), y(t), z(t), p(t), q(t))$$

of the characteristic equations is called *characteristic strip* or *Monge curve.*



GASPARD MONGE,
COMTE DE PÉLUSE 1746-1818

L'inventeur de
la géométrie descriptive

Gaspard MONGE,

• *Professeur de mathématiques et de physique à l'École du génie militaire de Mézières, il est invité par Turgot à venir à Paris en 1780, date à laquelle il entre à l'Académie des Sciences. Favorable à la Révolution, il devient ministre de la Marine en 1792. Il contribue ensuite à la création de l'École normale et à celle de l'École polytechnique. Il participe à la campagne d'Égypte et Napoléon le fait comte et sénateur. Son œuvre se caractérise par une vision globale, qui réunit les aspects analytique, géométrique et pratique.*

Figure 2.8: Gaspard Monge (Panthéon, Paris)

We will see, as in the quasilinear case, that the strips defined by the characteristic equations build the solution surface of the Cauchy initial value problem.

Let $z = u(x, y)$ be a solution of the general nonlinear differential equation (2.6).

Let (x_0, y_0, z_0) be fixed, then equation (2.6) defines a set of planes given by (x_0, y_0, z_0, p, q), i. e., planes given by $z = v(x, y)$ which contain the point (x_0, y_0, z_0) and for which $v_x = p$, $v_y = q$ at (x_0, y_0). In the case of quasilinear equations these set of planes is a bundle of planes which all contain a fixed straight line, see Section 2.1. In the general case of this section the situation is more complicated.

Consider the example

$$p^2 + q^2 = f(x, y, z), \tag{2.12}$$

where f is a given positive function. Let E be a plane defined by $z = v(x,y)$ and which contains (x_0, y_0, z_0). Then the normal on the plane E directed downward is

$$\mathbf{N} = \frac{1}{\sqrt{1 + |\nabla v|^2}}(p, q, -1),$$

where $p = v_x(x_0, y_0)$, $q = v_y(x_0, y_0)$. It follows from (2.12) that the normal $\mathbf{N}$ makes a constant angle with the z-axis, and the z-coordinate of $\mathbf{N}$ is constant, see Figure 2.9.

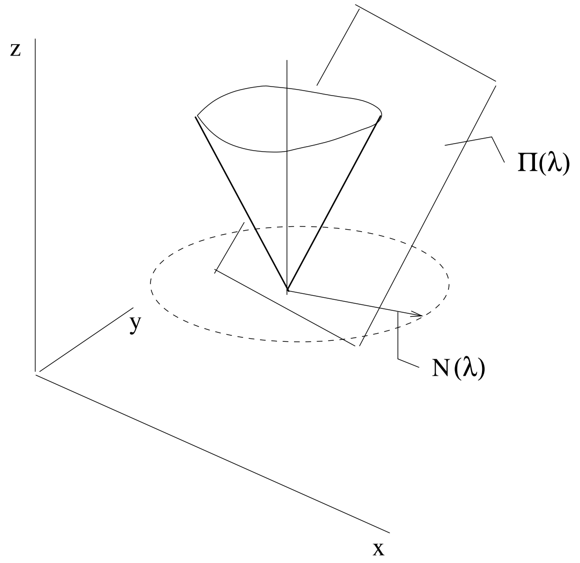


Figure 2.9: Monge cone in an example

Thus the endpoints of the normals fixed at (x_0, y_0, z_0) define a circle parallel to the (x, y)-plane, i. e., there is a cone which is the envelope of all these planes.

We assume that the general equation (2.6) defines such a Monge cone at each point in $\mathbb{R}^3$. Then we seek a surface S which touches at each point its Monge cone, see Figure 2.10.

More precisely, we assume there exists, as in the above example, a one parameter C^1-family

$$p(\lambda) = p(\lambda; x, y, z), \;\; q(\lambda) = q(\lambda; x, y, z)$$

of solutions of (2.6). These $(p(\lambda), q(\lambda))$ define a family $\Pi(\lambda)$ of planes.

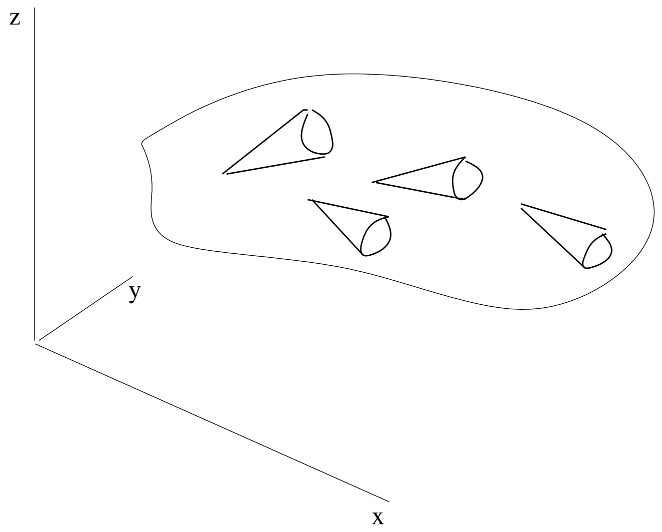


Figure 2.10: Monge cones

Let

$$\mathbf{x}(\tau) = (x(\tau), y(\tau), z(\tau))$$

be a curve on the surface S which touches at each point its Monge cone, see Figure 2.11. Thus we assume that at each point of the surface $\mathcal{S}$ the associated tangent plane coincides with a plane from the family $\Pi(\lambda)$ at this point. Consider the tangential plane $T_{\mathbf{x}_0}$ of the surface S at $\mathbf{x}_0 = (x(\tau_0), y(\tau_0), z(\tau_0))$. The straight line

$$\mathbf{l}(\sigma) = \mathbf{x}_0 + \sigma \mathbf{x}'(\tau_0), \qquad -\infty < \sigma < \infty,$$

is an apothem (in German: Mantellinie) of the cone by assumption and is contained in the tangential plane $T_{\mathbf{x}_0}$ as the tangent of a curve on the surface S. It is defined through

$$\mathbf{x}'(\tau_0) = \mathbf{l}'(\sigma). \tag{2.13}$$

The straight line $\mathbf{l}(\sigma)$ satisfies

$$l_3(\sigma) - z_0 = (l_1(\sigma) - x_0)p(\lambda_0) + (l_2(\sigma) - y_0)q(\lambda_0),$$

since it is contained in the tangential plane $T_{\mathbf{x}_0}$ defined by the slope (p, q). It follows

$$l_3'(\sigma) = p(\lambda_0)l_1'(\sigma) + q(\lambda_0)l_2'(\sigma).$$

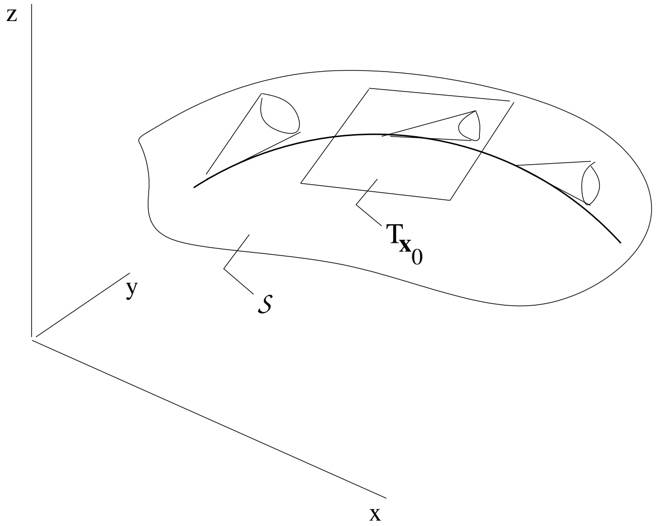


Figure 2.11: Monge cones along a curve on the surface

Together with (2.13) we obtain

$$z'(\tau) = p(\lambda_0)x'(\tau) + q(\lambda_0)y'(\tau). \tag{2.14}$$

The above straight line **l** is the limit of the intersection line of two neighbouring planes which envelopes the Monge cone:

$$\begin{aligned} z - z_0 &= (x - x_0)p(\lambda_0) + (y - y_0)q(\lambda_0) \\ z - z_0 &= (x - x_0)p(\lambda_0 + h) + (y - y_0)q(\lambda_0 + h). \end{aligned}$$

On the intersection one has

$$(x - x_0)p(\lambda) + (y - y_0)q(\lambda_0) = (x - x_0)p(\lambda_0 + h) + (y - y_0)q(\lambda_0 + h).$$

Let $h \to 0$, it follows

$$(x - x_0)p'(\lambda_0) + (y - y_0)q'(\lambda_0) = 0.$$

Since $x = l_1(\sigma)$, $y = l_2(\sigma)$ in this limit position, we have

$$p'(\lambda_0)l_1'(\sigma) + q'(\lambda_0)l_2'(\sigma) = 0,$$

and it follows from (2.13) that

$$p'(\lambda_0)x'(\tau) + q'(\lambda_0)y'(\tau) = 0. \tag{2.15}$$

From the differential equation $F(x_0,y_0,z_0,p(\lambda),q(\lambda))=0$ we see that

$$F_p p'(\lambda)+F_q q'(\lambda)=0. \tag{2.16}$$

Assume $x'(\tau_0)\neq 0$ and $F_p\neq 0$, then we obtain from (2.15), (2.16)

$$\frac{y'(\tau_0)}{x'(\tau_0)}=\frac{F_q}{F_p},$$

and from (2.14) (2.16) that

$$\frac{z'(\tau_0)}{x'(\tau_0)}=p+q\frac{F_q}{F_p}.$$

It follows, since τ_0 was an arbitrary fixed parameter,

$$\begin{aligned}\mathbf{x}'(\tau) &= (x'(\tau),y'(\tau),z'(\tau))\\ &= \left(x'(\tau),x'(\tau)\frac{F_q}{F_p},x'(\tau)\left(p+q\frac{F_q}{F_p}\right)\right)\\ &= \frac{x'(\tau)}{F_p}(F_p,F_q,pF_p+qF_q),\end{aligned}$$

i. e., the tangential vector $\mathbf{x}'(\tau)$ is proportional to (F_p,F_q,pF_p+qF_q). Set

$$a(\tau)=\frac{x'(\tau)}{F_p},$$

where $F=F(x(\tau),y(\tau),z(\tau),p(\lambda(\tau)),q(\lambda(\tau)))$. Introducing the new parameter t by the inverse of $\tau=\tau(t)$, where

$$t(\tau)=\int_{\tau_0}^{\tau} a(s)\ ds,$$

we obtain the characteristic equations (2.7)–(2.9). Here we denote $\mathbf{x}(\tau(t))$ by $\mathbf{x}(t)$ again. From the differential equation (2.6) and from (2.7)–(2.9) we get equations (2.10) and (2.11). Assume the surface $z=u(x,y)$ under consideration is in C^2, then

$$\begin{aligned}F_x+F_zp+F_pp_x+F_qp_y &= 0,\quad (q_x=p_y)\\ F_x+F_zp+x'(t)p_x+y'(t)p_y &= 0\\ F_x+F_zp+p'(t) &= 0\end{aligned}$$

since $p = p(x, y) = p(x(t), y(t))$ on the curve $\mathbf{x}(t)$. Thus equation (2.10) of the characteristic system is shown. Differentiating the differential equation (2.6) with respect to y, we get finally equation (2.11).

Remark. In the previous quasilinear case

$$F(x, y, z, p, q) = a_1(x, y, z)p + a_2(x, y, z)q - a_3(x, y, z)$$

the first three characteristic equations are the same:

$$x'(t) = a_1(x, y, z),\ \ y'(t) = a_2(x, y, z),\ \ z'(t) = a_3(x, y, z).$$

The point is that the right hand sides are independent on p or q. It follows from Theorem 2.1 that there exists a solution of the Cauchy initial value problem provided the initial data are noncharacteristic. That is, we do not need the other remaining two characteristic equations.

The other two equations (2.10) and (2.11) are satisfied in this quasilinear case automatically if there is a solution of the equation, see the above derivation of these equations.

The geometric meaning of the first three characteristic differential equations (2.7)–(2.11) is the following one. Each point of the curve $\mathcal{A}:\quad (x(t), y(t), z(t))$ corresponds a tangential plane with the normal direction $(-p, -q, 1)$ such that

$$z'(t) = p(t)x'(t) + q(t)y'(t).$$

This equation is called *strip condition.* On the other hand, let $z = u(x, y)$ defines a surface, then $z(t) := u(x(t), y(t))$ satisfies the strip condition, where $p = u_x$ and $q = u_y$, that is, the "scales" defined by the normals fit together.

Proposition 2.3. *$F(x, y, z, p, q)$ is an integral, i. e., it is constant along each characteristic curve.*

Proof.

$$\begin{aligned} \frac{d}{dt}F(x(t), y(t), z(t), p(t), q(t)) &= F_x x' + F_y y' + F_z z' + F_p p' + F_q q' \\ &= F_x F_p + F_y F_q + pF_z F_p + qF_z F_q \\ &\quad -F_p f_x - F_p F_z p - F_q F_y - F_q F_z q \\ &= 0. \end{aligned}$$

□

Corollary. Assume $F(x_0, y_0, z_0, p_0, q_0) = 0$, then $F = 0$ along characteristic curves with the initial data $(x_0, y_0, z_0, p_0, q_0)$.

Proposition 2.4. *Let $z = u(x, y)$, $u \in C^2$, be a solution of the nonlinear equation (2.6). Set*

$$z_0 = u(x_0, y_0,)\ p_0 = u_x(x_0, y_0),\ \ q_0 = u_y(x_0, y_0).$$

Then the associated characteristic strip is in the surface $\mathcal{S}$, defined by $z = u(x, y)$. Thus

$$\begin{aligned} z(t) &= u(x(t), y(t)) \\ p(t) &= u_x(x(t), y(t)) \\ q(t) &= u_y(x(t), y(t)), \end{aligned}$$

where $(x(t), y(t), z(t), p(t), q(t))$ is the solution of the characteristic system (2.7)–(2.11) with initial data $(x_0, y_0, z_0, p_0, q_0)$

Proof. Consider the initial value problem

$$\begin{aligned} x'(t) &= F_p(x, y, u(x, y), u_x(x, y), u_y(x, y)) \\ y'(t) &= F_q(x, y, u(x, y), u_x(x, y), u_y(x, y)) \end{aligned}$$

with the initial data $x(0) = x_0$, $y(0) = y_0$. We will show that

$$(x(t), y(t), u(x(t), y(t)), u_x(x(t), y(t)), u_y(x(t), y(t)))$$

is a solution of the characteristic system. We recall that the solution exists and is uniquely determined.

Set $z(t) = u(x(t), y(t))$, then $(x(t), y(t), z(t)) \subset \mathcal{S}$, and

$$z'(t) = u_x x'(t) + u_y y'(t) = u_x F_p + u_y F_q.$$

Set $p(t) = u_x(x(t), y(t)),\ \ q(t) = u_y(x(t), y(t))$, then

$$\begin{aligned} p'(t) &= u_{xx} F_p + u_{xy} F_q \\ q'(t) &= u_{yx} F_p + u_{yy} F_q. \end{aligned}$$

Finally, from the differential equation $F(x, y, u(x, y), u_x(x, y), u_y(x, y)) = 0$ it follows

$$\begin{aligned} p'(t) &= -F_x - F_u p \\ q'(t) &= -F_y - F_u q. \end{aligned}$$

□

2.3.1 Initial value problem of Cauchy

Let

$$x=x_0(s),\ y=y_0(s),\ z=z_0(s),\ p=p_0(s),\ q=q_0(s),\ s_1<s<s_2, \tag{2.17}$$

be a given *initial strip* such that the *strip condition*

$$z_0'(s)=p_0(s)x_0'(s)+q_0(s)y_0'(s) \tag{2.18}$$

is satisfied. Moreover, we assume that the initial strip satisfies the nonlinear equation, that is,

$$F(x_0(s),y_0(s),z_0(s),p_0(s),q_0(s))=0. \tag{2.19}$$

Initial value problem of Cauchy: *Find a C^2-solution $z=u(x,y)$ of $F(x,y,z,p,q)=0$ such that the surface $\mathcal{S}$ defined by $z=u(x,y)$ contains the above initial strip.*

Similar to the quasilinear case we will show that the set of strips defined by the characteristic system which are sticked at the initial strip, see Figure 2.12, fit together and define the surface for which we are looking at.

Definition. A strip $(x(\tau),y(\tau),z(\tau),p(\tau),q(\tau))$, $\tau_1<\tau<\tau_2$, is said to be *noncharacteristic* if

$$x'(\tau)F_q(x(\tau),y(\tau),z(\tau),p(\tau),q(\tau))-y'(\tau)F_p(x(\tau),y(\tau),z(\tau),p(\tau),q(\tau))\neq 0.$$

Theorem 2.2. *For a given noncharacteristic initial strip (2.17), $x_0,\ y_0,\ z_0\in C^2$ and $p_0,\ q_0\in C^1$ which satisfies the strip condition (2.18) and the differential equation (2.19) there exists exactly one solution $z=u(x,y)$ of the Cauchy initial value problem in a neighbourhood of the initial curve $(x_0(s),y_0(s),z_0(s))$, i. e., $z=u(x,y)$ is the solution of the differential equation (2.6) and $u(x_0(s),y_0(s))=z_0(s)$, $u_x(x_0(s),y_0(s))=p_0(s)$, $u_y(x_0(s),y_0(s))=q_0(s)$.*

Proof. Consider the system (2.7)–(2.11) with initial data

$$x(s,0)=x_0(s),\ y(s,0)=y_0(s),\ z(s,0)=z_0(s),\ p(s,0)=p_0(s),\ q(s,0)=q_0(s).$$

We will show that the surface defined by $x=x(s,t),\ y(s,t)$ is the surface defined by $z=u(x,y)$, where u is the solution of the Cauchy initial value

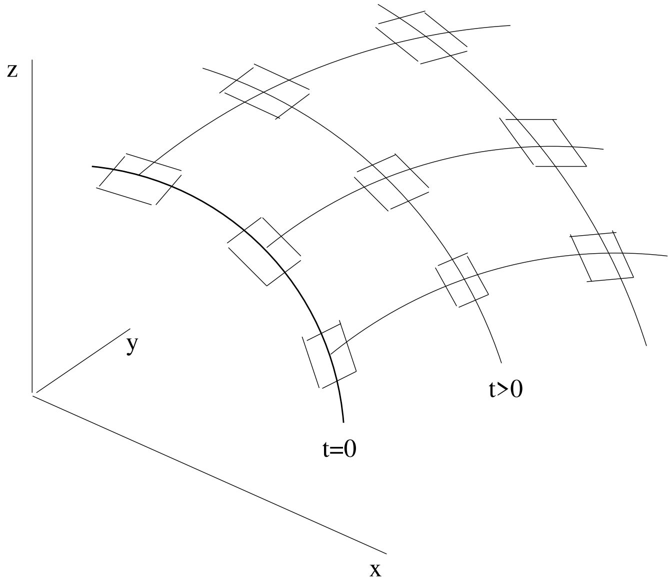


Figure 2.12: Construction of the solution

problem. It turns out that $u(x, y) = z(s(x, y), t(x, y))$, where $s = s(x, y)$, $t = t(x, y)$ is the inverse of $x = x(s, t)$, $y = y(s, t)$ in a neigbourhood of $t = 0$. This inverse exists since the initial strip is noncharacteristic by assumption:

$$\det \frac{\partial(x, y)}{\partial(s, t)}\Big|_{t=0} = x_0 F_q - y_0 F_q \neq 0.$$

Set

$$P(x, y) = p(s(x, y), t(x, y)), \quad Q(x, y) = q(s(x, y), t(x, y)).$$

From Proposition 2.3 and Proposition 2.4 it follows $F(x, y, u, P, Q) = 0$. We will show that $P(x, y) = u_x(x, y)$ and $Q(x, y) = u_y(x, y)$. To see this, we consider the function

$$h(s, t) = z_s - px_s - qy_s.$$

One has

$$h(s, 0) = z_0'(s) - p_0(s)x_0'(s) - q_0(s)y_0'(s) = 0$$

since the initial strip satisfies the strip condition by assumption. In the following we will find that for fixed s the function h satisfies a linear homogeneous ordininary differential equation of first order. Consequently,

$h(s,t)=0$ in a neighbourhood of $t=0$. Thus the strip condition is also satisfied along strips transversally to the characteristic strips, see Figure 2.18. Thaen the set of "scales" fit together and define a surface like the scales of a fish.

From the definition of $h(s,t)$ and the characteristic equations we get

$$\begin{aligned} h_t(s,t) &= z_{st}-p_tx_s-q_ty_s-px_{st}-qy_{st} \\ &= \frac{\partial}{\partial s}(z_t-px_t-qy_t)+p_sx_t+q_sy_t-q_ty_s-p_tx_s \\ &= (px_s+qy_s)F_z+F_xx_s+F_yz_s+F_pp_s+F_qq_s. \end{aligned}$$

Since $F(x(s,t),y(s,t),z(s,t),p(s,t),q(s,t))=0$, it follows after differentiation of this equation with respect to s the differential equation

$$h_t=-F_zh.$$

Hence $h(s,t)\equiv 0$, since $h(s,0)=0$.

Thus we have

$$\begin{aligned} z_s &= px_s+qy_s \\ z_t &= px_t+qy_t \\ z_s &= u_xx_s+u_yy_s \\ z_t &= u_xy_t+u_yy_t. \end{aligned}$$

The first equation was shown above, the second is a characteristic equation and the last two follow from $z(s,t)=u(x(s,t),y(s,t))$. This system implies

$$\begin{aligned} (P-u_x)x_s+(Q-u_y)y_s &= 0 \\ (P-u_x)x_t+(Q-u_y)y_t &= 0. \end{aligned}$$

It follows $P=u_x$ and $Q=u_y$.

The initial conditions

$$\begin{aligned} u(x(s,0),y(s,0)) &= z_0(s) \\ u_x(x(s,0),y(s,0)) &= p_0(s) \\ u_y(x(s,0),y(s,0)) &= q_0(s) \end{aligned}$$

are satisfied since

$$\begin{aligned} u(x(s,t),y(s,t)) &= z(s(x,y),t(x,y))=z(s,t) \\ u_x(x(s,t),y(s,t)) &= p(s(x,y),t(x,y))=p(s,t) \\ u_y(x(s,t),y(s,t)) &= q(s(x,y),t(x,y))=q(s,t). \end{aligned}$$

The uniqueness follows as in the proof of Theorem 2.1. □

Example. A differential equation which occurs in the geometrical optic is

$$u_x^2+u_y^2=f(x,y),$$

where the positive function $f(x,y)$ is the index of refraction. The level sets defined by $u(x,y)=const.$ are called *wave fronts*. The characteristic curves $(x(t),y(t))$ are the rays of light. If n is a constant, then the rays of light are straight lines. In $\mathbb{R}^3$ the equation is

$$u_x^2+u_y^2+u_z^2=f(x,y,z).$$

Thus we have to extend the previous theory from $\mathbb{R}^2$ to $\mathbb{R}^n$, $n\ge 3$.

2.4 Nonlinear equations in $\mathbb{R}^n$

Here we consider the nonlinear differential equation

$$F(x,z,p)=0, \tag{2.20}$$

where

$$x=(x_1,\ldots,x_n),\ z=u(x):\ \Omega\subset\mathbb{R}^n\mapsto\mathbb{R},\ p=\nabla u.$$

The following system of $2n+1$ ordinary differential equations is called *characteristic system*.

$$\begin{aligned}x'(t)&=\nabla_pF\\ z'(t)&=p\cdot\nabla_pF\\ p'(t)&=-\nabla_xF-F_zp.\end{aligned}$$

Let

$$x_0(s)=(x_{01}(s),\ldots,x_{0n}(s)),\ s=(s_1,\ldots,s_{n-1}),$$

be a given regular (n-1)-dimensional C^2-hypersurface in $\mathbb{R}^n$, i. e., we assume

$$\text{rank}\frac{\partial x_0(s)}{\partial s}=n-1.$$

Here $s\in D$ is a parameter from an $(n-1)$-dimensional parameter domain D.

For example, $x=x_0(s)$ defines in the three dimensional case a regular surface in $\mathbb{R}^3$.

Assume

$$z_0(s):\ D \mapsto \mathbb{R},\ p_0(s) = (p_{01}(s), \ldots, p_{0n}(s))$$

are given sufficiently regular functions.

The $(2n+1)$-vector

$$(x_0(s), z_0(s), p_0(s))$$

is called *initial strip manifold* and the condition

$$\frac{\partial z_0}{\partial s_l} = \sum_{i=1}^{n-1} p_{0i}(s) \frac{\partial x_{0i}}{\partial s_l},$$

$l = 1, \ldots, n-1$, *strip condition.*

The initial strip manifold is said to be *noncharacteristic* if

$$\det \begin{pmatrix} F_{p_1} & F_{p_2} & \cdots & F_{p_n} \\ \frac{\partial x_{01}}{\partial s_1} & \frac{\partial x_{02}}{\partial s_1} & \cdots & \frac{\partial x_{0n}}{\partial s_1} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial x_{01}}{\partial s_{n-1}} & \frac{\partial x_{02}}{\partial s_{n-1}} & \cdots & \frac{\partial x_{0n}}{\partial s_{n-1}} \end{pmatrix} \neq 0,$$

where the argument of F_{p_j} is the initial strip manifold.

Initial value problem of Cauchy. *Seek a solution* $z = u(x)$ *of the differential equation (2.20) such that the initial manifold is a subset of* $\{(x, u(x), \nabla u(x)) :\ x \in \Omega\}$.

As in the two dimensional case we have under additional regularity assumptions

Theorem 2.3. *Suppose the initial strip manifold is not characteristic and satisfies differential equation (2.20), that is,* $F(x_0(s), z_0(s), p_0(s)) = 0$. *Then there is a neighbourhood of the initial manifold* $(x_0(s), z_0(s))$ *such that there exists a unique solution of the Cauchy initial value problem.*

Sketch of proof. Let

$$x = x(s,t),\ z = z(s,t),\ p = p(s,t)$$

be the solution of the characteristic system and let

$$s = s(x),\ t = t(x)$$

be the inverse of $x = x(s,t)$ which exists in a neighbourhood of $t = 0$. Then, it turns out that

$$z = u(x) := z(s_1(x_1,\ldots,x_n),\ldots,s_{n-1}(x_1,\ldots,x_n),t(x_1,\ldots,x_n))$$

is the solution of the problem.

2.5 Hamilton-Jacobi theory

The nonlinear equation (2.20) of previous section in one more dimension is

$$F(x_1,\ldots,x_n,x_{n+1},z,p_1,\ldots,p_n,p_{n+1}) = 0.$$

The content of the Hamilton[1]-Jacobi[2] theory is the theory of the special case

$$F \equiv p_{n+1} + H(x_1,\ldots,x_n,x_{n+1},p_1,\ldots,p_n) = 0, \tag{2.21}$$

i. e., the equation is linear in p_{n+1} and does not depend on z explicitly.

Remark. Formally, one can write equation (2.20)

$$F(x_1,\ldots,x_n,u,u_{x_1},\ldots,u_{x_n}) = 0$$

as an equation of type (2.21). Set $x_{n+1} = u$ and seek u implicitely from

$$\phi(x_1,\ldots,x_n,x_{n+1}) = const.,$$

where ϕ is a function which is defined by a differential equation.
Assume $\phi_{x_{n+1}} \neq 0$, then

$$\begin{aligned}
0 &= F(x_1,\ldots,x_n,u,u_{x_1},\ldots,u_{x_n}) \\
&= F(x_1,\ldots,x_n,x_{n+1},-\frac{\phi_{x_1}}{\phi_{x_{n+1}}},\ldots,-\frac{\phi_{x_n}}{\phi_{x_{n+1}}}) \\
&= :G(x_1,\ldots,x_{n+1},\phi_1,\ldots,\phi_{x_{n+1}}).
\end{aligned}$$

Suppose that $G_{\phi_{x_{n+1}}} \neq 0$, then

$$\phi_{x_{n+1}} = H(x_1,\ldots,x_n,x_{n+1},\phi_{x_1},\ldots,\phi_{x_{n+1}}).$$

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[1] Hamilton, William Rowan, 1805–1865

[2] Jacobi, Carl Gustav, 1805–1851

The associated characteristic equations to (2.21) are

$$\begin{aligned}
x_{n+1}'(\tau) &= F_{p_{n+1}} = 1\\
x_k'(\tau) &= F_{p_k} = H_{p_k}, \qquad k = 1, \ldots, n\\
z'(\tau) &= \sum_{l=1}^{n+1} p_l F_{p_l} = \sum_{l=1}^{n} p_l H_{p_l} + p_{n+1}\\
&= \sum_{l=1}^{n} p_l H_{p_l} - H\\
p_{n+1}'(\tau) &= -F_{x_{n+1}} - F_z p_{n+1}\\
&= -F_{x_{n+1}}\\
p_k'(\tau) &= -F_{x_k} - F_z p_k\\
&= -F_{x_k}, \qquad k = 1, \ldots, n.
\end{aligned}$$

Set $t := x_{n+1}$, then we can write partial differential equation (2.21) as

$$u_t + H(x, t, \nabla_x u) = 0 \tag{2.22}$$

and $2n$ of the characteristic equations are

$$x'(t) = \nabla_p H(x, t, p) \tag{2.23}$$
$$p'(t) = -\nabla_x H(x, t, p). \tag{2.24}$$

Here is

$$x = (x_1, \ldots, x_n),\ p = (p_1, \ldots, p_n).$$

Let $x(t),\ p(t)$ be a solution of (2.23) and (2.24), then it follows $p_{n+1}'(t)$ and $z'(t)$ from the characteristic equations

$$\begin{aligned}
p_{n+1}'(t) &= -H_t\\
z'(t) &= p \cdot \nabla_p H - H.
\end{aligned}$$

Definition. The function $H(x, t, p)$ is called *Hamilton function*, equation (2.21) *Hamilton-Jacobi equation* and the system (2.23), (2.24) *canonical system to H.*

There is an interesting interplay between the Hamilton-Jacobi equation and the canonical system. According to the previous theory we can construct a solution of the Hamilton-Jacobi equation by using solutions of the canonical system. On the other hand, one obtains from solutions of the Hamilton-Jacobi equation also solutions of the canonical system of ordinary differential equations.

Definition. A solution $\phi(a; x, t)$ of the Hamilton-Jacobi equation, where $a = (a_1, \ldots, a_n)$ is an n-tuple of real parameters, is called a *complete integral* of the Hamilton-Jacobi equation if

$$\det(\phi_{x_i a_l})_{i,l=1}^n \neq 0.$$

Remark. If u is a solution of the Hamilton-Jacobi equation, then also $u + const.$

Theorem 2.4 (Jacobi). *Assume*

$$u = \phi(a; x, t) + c, \;\; c = const., \;\; \phi \in C^2 \;\; \textit{in its arguments},$$

is a complete integral. Then one obtains by solving of

$$b_i = \phi_{a_i}(a; x, t)$$

with respect to $x_l = x_l(a, b, t)$, *where* b_i $i = 1, \ldots, n$ *are given real constants, and then by setting*

$$p_k = \phi_{x_k}(a; x(a, b; t), t)$$

a 2n-parameter family of solutions of the canonical system.

Proof. Let

$$x_l(a, b; t), \;\; l = 1, \ldots, n,$$

be the solution of the above system. The solution exists since ϕ is a complete integral by assumption. Set

$$p_k(a, b; t) = \phi_{x_k}(a; x(a, b; t), t), \;\; k = 1, \ldots, n.$$

We will show that x and p solves the canonical system. Differentiating $\phi_{a_i} = b_i$ with respect to t and the Hamilton-Jacobi equation $\phi_t + H(x, t, \nabla_x \phi) = 0$ with respect to a_i, we obtain for $i = 1, \ldots, n$

$$\begin{aligned} \phi_{ta_i} + \sum_{k=1}^n \phi_{x_k a_i} \frac{\partial x_k}{\partial t} &= 0 \\ \phi_{ta_i} + \sum_{k=1}^n \phi_{x_k a_i} H p_k &= 0. \end{aligned}$$

Since ϕ is a complete integral it follows for $k = 1, \ldots, n$

$$\frac{\partial x_k}{\partial t} = H_{p_k}.$$

Along a trajectory, i. e., where $a,\ b$ are fixed, it is $\frac{\partial x_k}{\partial t} = x_k'(t)$. Thus

$$x_k'(t) = H_{p_k}.$$

Now we differentiate $p_i(a,b;t)$ with respect to t and $\phi_t + H(x,t,\nabla_x\phi) = 0$ with respect to x_i, and obtain

$$\begin{aligned} p_i'(t) &= \phi_{x_it} + \sum_{k=1}^n \phi_{x_ix_k} x_k'(t) \\ 0 &= \phi_{x_it} + \sum_{k=1}^n \phi_{x_ix_k} H_{p_k} + H_{x_i} \\ 0 &= \phi_{x_it} + \sum_{k=1}^n \phi_{x_ix_k} x_k'(t) + H_{x_i} \end{aligned}$$

It follows finally that $p_i'(t) = -H_{x_i}$. □

Example: **Kepler problem**

The motion of a mass point in a central field takes place in a plane, say the (x,y)-plane, see Figure 2.13, and satisfies the system of ordinary differential equations of second order

$$x''(t) = U_x,\ y''(t) = U_y,$$

where

$$U(x,y) = \frac{k^2}{\sqrt{x^2+y^2}}.$$

Here we assume that k^2 is a positive constant and that the mass point is attracted of the origin. In the case that it is pushed one has to replace U by $-U$. See Landau and Lifschitz [12], Vol 1, for example, for the related physics.

Set

$$p = x',\ q = y'$$

and

$$H = \frac{1}{2}(p^2+q^2) - U(x,y),$$

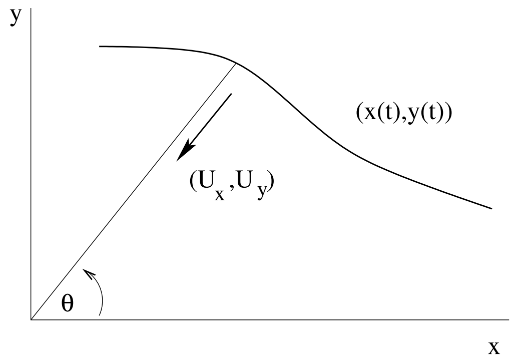

Figure 2.13: Motion in a central field

then

$$\begin{aligned} x'(t) &= H_p,\ y'(t) = H_q \\ p'(t) &= -H_x,\ q'(t) = -H_y. \end{aligned}$$

The associated Hamilton-Jacobi equation is

$$\phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2) = \frac{k^2}{\sqrt{x^2+y^2}}.$$

which is in polar coordinates (r, θ)

$$\phi_t + \frac{1}{2}(\phi_r^2 + \frac{1}{r^2}\phi_\theta^2) = \frac{k^2}{r}. \tag{2.25}$$

Now we will seek a complete integral of (2.25) by making the ansatz

$$\phi_t = -\alpha = const. \;\; \phi_\theta = -\beta = const. \tag{2.26}$$

and obtain from (2.25) that

$$\phi = \pm \int_{r_0}^{r} \sqrt{2\alpha + \frac{2k^2}{\rho} - \frac{\beta^2}{\rho^2}}\, d\rho + c(t, \theta).$$

From ansatz (2.26) it follows

$$c(t, \theta) = -\alpha t - \beta\theta.$$

Therefore we have a two parameter family of solutions

$$\phi = \phi(\alpha, \beta; \theta, r, t)$$

of the Hamilton-Jacobi equation. This solution is a complete integral, see an exercise. According to the theorem of Jacobi set

$$\phi_\alpha = -t_0,\quad \phi_\beta = -\theta_0.$$

Then

$$t - t_0 = -\int_{r_0}^{r} \frac{d\rho}{\sqrt{2\alpha + \frac{2k^2}{\rho} - \frac{\beta^2}{\rho^2}}}.$$

The inverse function $r = r(t)$, $r(0) = r_0$, is the r-coordinate depending on time t, and

$$\theta - \theta_0 = \beta \int_{r_0}^{r} \frac{d\rho}{\rho^2\sqrt{2\alpha + \frac{2k^2}{\rho} - \frac{\beta^2}{\rho^2}}}.$$

Substitution $\tau = \rho^{-1}$ yields

$$\begin{aligned}\theta - \theta_0 &= -\beta \int_{1/r_0}^{1/r} \frac{d\tau}{\sqrt{2\alpha + 2k^2\tau - \beta^2\tau^2}}\\ &= -\arcsin\left(\frac{\frac{\beta^2}{k^2}\frac{1}{r} - 1}{\sqrt{1 + \frac{2\alpha\beta^2}{k^4}}}\right) + \arcsin\left(\frac{\frac{\beta^2}{k^2}\frac{1}{r_0} - 1}{\sqrt{1 + \frac{2\alpha\beta^2}{k^4}}}\right).\end{aligned}$$

Set

$$\theta_1 = \theta_0 + \arcsin\left(\frac{\frac{\beta^2}{k^2}\frac{1}{r_0} - 1}{\sqrt{1 + \frac{2\alpha\beta^2}{k^4}}}\right)$$

and

$$p = \frac{\beta^2}{k^2},\quad \epsilon^2 = \sqrt{1 + \frac{2\alpha\beta^2}{k^4}},$$

then

$$\theta - \theta_1 = -\arcsin\left(\frac{\frac{p}{r} - 1}{\epsilon^2}\right).$$

It follows

$$r = r(\theta) = \frac{p}{1 - \epsilon^2 \sin(\theta - \theta_1)},$$

which is the polar equation of conic sections. It defines an ellipse if $0 \le \epsilon < 1$, a parabola if $\epsilon = 1$ and a hyperbola if $\epsilon > 1$, see Figure 2.14 for the case of an ellipse, where the origin of the coordinate system is one of the focal points of the ellipse.

For another application of the Jacobi theorem see Courant and Hilbert [4], Vol. 2, pp. 94, where geodedics on an ellipsoid are studied.

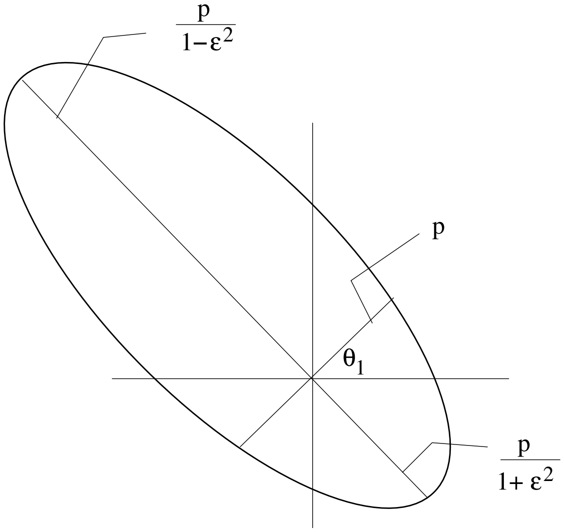

Figure 2.14: The case of an ellipse

2.6 Exercises

1. Suppose $u:\ \mathbb{R}^2 \mapsto \mathbb{R}$ is a solution of

   $$a(x,y)u_x + b(x,y)u_y = 0.$$

   Show that for arbitrary $H \in C^1$ also $H(u)$ is a solution.

2. Find a solution $u \not\equiv const.$ of

   $$u_x + u_y = 0$$

   such that

   $$\text{graph}(u) := \{(x,y,z) \in \mathbb{R}^3 :\ z = u(x,y),\ (x,y) \in \mathbb{R}^2\}$$

   contains the straight line $(0,0,1) + s(1,1,0),\ s \in \mathbb{R}$.

3. Let $\phi(x,y)$ be a solution of

   $$a_1(x,y)u_x + a_2(x,y)u_y = 0\ .$$

   Prove that level curves $S_C := \{(x,y) :\ \phi(x,y) = C = const.\}$ are characteristic curves, provided that $\nabla\phi \neq 0$ and $(a_1,a_2) \neq (0,0)$.

4. Prove Proposition 2.2.

5. Find two different solutions of the initial value problem

$$u_x + u_y = 1,$$

where the initial data are $x_0(s) = s,\ y_0(s) = s,\ z_0(s) = s$.

Hint: (x_0, y_0) is a characteristic curve.

6. Solve the initial value problem

$$xu_x + yu_y = u$$

with initial data $x_0(s) = s,\ y_0(s) = 1,\ z_0(s)$, where z_0 is given.

7. Solve the initial value problem

$$-xu_x + yu_y = xu^2,$$

$x_0(s) = s,\ y_0(s) = 1,\ z_0(s) = \mathrm{e}^{-s}$.

8. Solve the initial value problem

$$uu_x + u_y = 1,$$

$x_0(s) = s,\ y_0(s) = s,\ z_0(s) = s/2$ if $0 < s < 1$.

9. Solve the initial value problem

$$uu_x + uu_y = 2,$$

$x_0(s) = s,\ y_0(s) = 1,\ z_0(s) = 1 + s$ if $0 < s < 1$.

10. Solve the initial value problem $u_x^2 + u_y^2 = 1 + x$ with given initial data $x_0(s) = 0,\ y_0(s) = s,\ u_0(s) = 1,\ p_0(s) = 1,\ q_0(s) = 0,\ -\infty < s < \infty$.

11. Find the solution $\Phi(x, y)$ of

$$(x - y)u_x + 2yu_y = 3x$$

such that the surface defined by $z = \Phi(x, y)$ contains the curve

$$C:\quad x_0(s) = s,\ y_0(s) = 1,\ z_0(s) = 0,\ s \in \mathbb{R}.$$

12. Solve the following initial problem of chemical kinetics.

$$u_x + u_y = \left(k_0 \mathrm{e}^{-k_1 x} + k_2\right)(1-u)^2,\ x > 0,\ y > 0$$

with the initial data $u(x,0) = 0,\ u(0,y) = u_0(y)$, where u_0, $0 < u_0 < 1$, is given.

13. Solve the Riemann problem

$$\begin{aligned} u_{x_1} + u_{x_2} &= 0 \\ u(x_1,0) &= g(x_1) \end{aligned}$$

in $\Omega_1 = \{(x_1,x_2) \in \mathbb{R}^2 :\ x_1 > x_2\}$ and in $\Omega_2 = \{(x_1,x_2) \in \mathbb{R}^2 :\ x_1 < x_2\}$, where

$$g(x_1) = \begin{cases} u_l & : \ x_1 < 0 \\ u_r & : \ x_1 > 0 \end{cases}$$

with constants $u_l \neq u_r$.

14. Determine the opening angle of the Monge cone, i. e., the angle between the axis and the apothem (in German: Mantellinie) of the cone, for equation

$$u_x^2 + u_y^2 = f(x,y,u),$$

where $f > 0$.

15. Solve the initial value problem

$$u_x^2 + u_y^2 = 1,$$

where $x_0(\theta) = a\cos\theta,\ y_0(\theta) = a\sin\theta,\ z_0(\theta) = 1,\ p_0(\theta) = \cos\theta,\ q_0(\theta) = \sin\theta$ if $0 \le \theta < 2\pi$, $a = const. > 0$.

16. Show that the integral $\phi(\alpha,\beta;\theta,r,t)$, see the Kepler problem, is a complete integral.

17. a) Show that $S = \sqrt{\alpha}\ x + \sqrt{1-\alpha}\ y + \beta$, $\alpha,\ \beta \in \mathbb{R},\ 0 < \alpha < 1$, is a complete integral of $S_x - \sqrt{1 - S_y^2} = 0$.

    b) Find the envelope of this family of solutions.

18. Determine the length of the half axis of the ellipse

$$r = \frac{p}{1 - \varepsilon^2 \sin(\theta - \theta_0)},\ 0 \le \varepsilon < 1.$$

19. Find the Hamilton function $H(x,p)$ of the Hamilton-Jacobi-Bellman differential equation if $h=0$ and $f=Ax+B\alpha$, where A, B are constant and real matrices, $A: \mathbb{R}^m \mapsto \mathbb{R}^n$, B is an orthogonal real $n \times n$-Matrix and $p \in \mathbb{R}^n$ is given. The set of admissible controls is given by

$$U = \{\alpha \in \mathbb{R}^n : \sum_{i=1}^{n} \alpha_i^2 \leq 1\} .$$

Remark. The Hamilton-Jacobi-Bellman equation is formally the Hamilton-Jacobi equation $u_t + H(x, \nabla u) = 0$, where the Hamilton function is defined by

$$H(x,p) := \min_{\alpha \in U} \left(f(x,\alpha) \cdot p + h(x,\alpha) \right),$$

$f(x,\alpha)$ and $h(x,\alpha)$ are given. See for example, Evans [5], Chapter 10.

Chapter 3

Classification

Different types of problems in physics, for example, correspond different types of partial differential equations. The methods how to solve these equations differ from type to type.

The classification of differential equations follows from one single question: Can we calculate formally the solution if sufficiently many initial data are given? Consider the initial problem for an ordinary differential equation $y'(x) = f(x, y(x))$, $y(x_0) = y_0$. Then one can determine formally the solution, provided the function $f(x, y)$ is sufficiently regular. The solution of the initial value problem is formally given by a power series. This formal solution is a solution of the problem if $f(x, y)$ is real analytic according to a theorem of Cauchy. In the case of partial differential equations the related theorem is the Theorem of Cauchy-Kowalevskaya. Even in the case of ordinary differential equations the situation is more complicated if y' is implicitly defined, i. e., the differential equation is $F(x, y(x), y'(x)) = 0$ for a given function F.

3.1 Linear equations of second order

The general nonlinear partial differential equation of second order is

$$F(x, u, Du, D^2u) = 0,$$

where $x \in \mathbb{R}^n$, $u : \ \Omega \subset \mathbb{R}^n \mapsto \mathbb{R}$, $Du \equiv \nabla u$ and D^2u stands for all second derivatives. The function F is given and sufficiently regular with respect to its $2n + 1 + n^2$ arguments.

In this section we consider the case

$$\sum_{i,k=1}^{n} a^{ik}(x)u_{x_i x_k} + f(x,u,\nabla u) = 0. \tag{3.1}$$

The equation is *linear* if

$$f = \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u + d(x).$$

Concerning the classification the *main part*

$$\sum_{i,k=1}^{n} a^{ik}(x)u_{x_i x_k}$$

plays the essential role. Suppose $u \in C^2$, then we can assume, without restriction of generality, that $a^{ik} = a^{ki}$, since

$$\sum_{i,k=1}^{n} a^{ik}u_{x_i x_k} = \sum_{i,k=1}^{n} (a^{ik})^\star u_{x_i x_k},$$

where

$$(a^{ik})^\star = \frac{1}{2}(a^{ik} + a^{ki}).$$

Consider a hypersurface $\mathcal{S}$ in $\mathbb{R}^n$ defined implicitly by $\chi(x) = 0$, $\nabla\chi \neq 0$, see Figure 3.1

Assume u and ∇u are given on $\mathcal{S}$.

Problem: *Can we calculate all other derivatives of u on $\mathcal{S}$ by using differential equation (3.1) and the given data?*

We will find an answer if we map $\mathcal{S}$ onto a hyperplane $\mathcal{S}_0$ by a mapping

$$\begin{aligned}\lambda_n &= \chi(x_1,\ldots,x_n)\\ \lambda_i &= \lambda_i(x_1,\ldots,x_n),\ i = 1,\ldots,n-1,\end{aligned}$$

for functions λ_i such that

$$\det\frac{\partial(\lambda_1,\ldots,\lambda_n)}{\partial(x_1,\ldots,x_n)} \neq 0$$

in $\Omega \subset \mathbb{R}^n$. It is assumed that χ and λ_i are sufficiently regular. Such a mapping $\lambda = \lambda(x)$ exists, see an exercise.

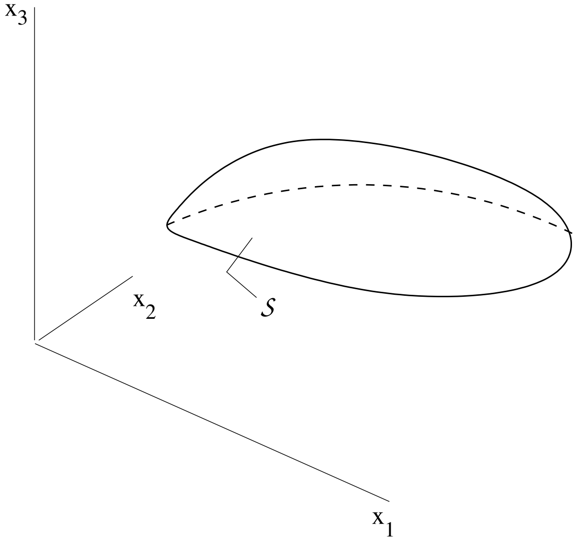

Figure 3.1: Initial manifold $\mathcal{S}$

The above transform maps $\mathcal{S}$ onto a subset of the hyperplane defined by $\lambda_n = 0$, see Figure 3.2.

We will write the differential equation in these new coordinates. Here we use Einstein's convention, i. e., we add terms with repeating indices. Since

$$u(x) = u(x(\lambda)) =: v(\lambda) = v(\lambda(x)),$$

where $x = (x_1, \ldots, x_n)$ and $\lambda = (\lambda_1, \ldots, \lambda_n)$, we get

$$\begin{aligned} u_{x_j} &= v_{\lambda_i} \frac{\partial \lambda_i}{\partial x_j}, \qquad (3.2) \\ u_{x_j x_k} &= v_{\lambda_i \lambda_l} \frac{\partial \lambda_i}{\partial x_j} \frac{\partial \lambda_l}{\partial x_k} + v_{\lambda_i} \frac{\partial^2 \lambda_i}{\partial x_j \partial x_k}. \end{aligned}$$

Thus, differential equation (3.1) in the new coordinates is given by

$$a^{jk}(x) \frac{\partial \lambda_i}{\partial x_j} \frac{\partial \lambda_l}{\partial x_k} v_{\lambda_i \lambda_l} + \text{terms known on } \mathcal{S}_0 = 0.$$

Since $v_{\lambda_k}(\lambda_1, \ldots, \lambda_{n-1}, 0)$, $k = 1, \ldots, n$, are known, see (3.2), it follows that $v_{\lambda_k \lambda_l}$, $l = 1, \ldots, n-1$, are known on $\mathcal{S}_0$. Thus we know all second derivatives $v_{\lambda_i \lambda_j}$ on $\mathcal{S}_0$ with the only exception of $v_{\lambda_n \lambda_n}$.

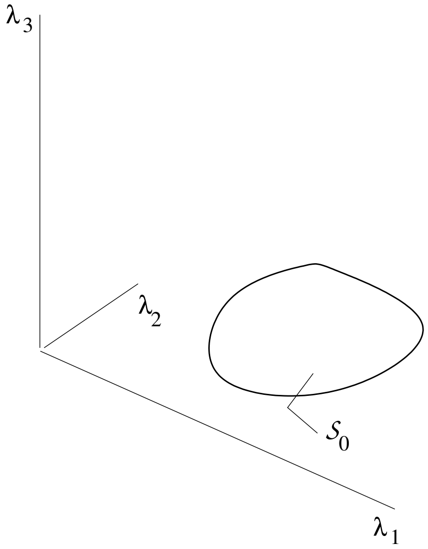

Figure 3.2: Transformed flat manifold $\mathcal{S}_0$

We recall that, provided v is sufficiently regular,

$$v_{\lambda_k\lambda_l}(\lambda_1, \ldots, \lambda_{n-1}, 0)$$

is the limit of

$$\frac{v_{\lambda_k}(\lambda_1, \ldots, \lambda_l + h, \lambda_{l+1}, \ldots, \lambda_{n-1}, 0) - v_{\lambda_k}(\lambda_1, \ldots, \lambda_l, \lambda_{l+1}, \ldots, \lambda_{n-1}, 0)}{h}$$

as $h \to 0$.

Thus the differential equation can be written as

$$\sum_{j,k=1}^{n} a^{jk}(x) \frac{\partial \lambda_n}{\partial x_j} \frac{\partial \lambda_n}{\partial x_k} v_{\lambda_n\lambda_n} = \text{terms known on } \mathcal{S}_0.$$

It follows that we can calculate $v_{\lambda_n\lambda_n}$ if

$$\sum_{i,j=1}^{n} a^{ij}(x) \chi_{x_i} \chi_{x_j} \neq 0 \tag{3.3}$$

on $\mathcal{S}$. This is a condition for the given equation and for the given surface $\mathcal{S}$.

Definition. The differential equation

$$\sum_{i,j=1}^{n} a^{ij}(x)\chi_{x_i}\chi_{x_j} = 0$$

is called *characteristic differential equation* associated to the given differential equation (3.1).

If χ, $\nabla\chi \neq 0$, is a solution of the characteristic differential equation, then the surface defined by $\chi = 0$ is called *characteristic surface.*

Remark. The condition (3.3) is satisfied for each χ with $\nabla\chi \neq 0$ if the quadratic matrix $(a^{ij}(x))$ is positive or negative definite for each $x \in \Omega$, which is equivalent to the property that all eigenvalues are different from zero and have the same sign. This follows since there is a $\lambda(x) > 0$ such that, in the case that the matrix (a^{ij}) is poitive definite,

$$\sum_{i,j=1}^{n} a^{ij}(x)\zeta_i\zeta_j \geq \lambda(x)|\zeta|^2$$

for all $\zeta \in \mathbb{R}^n$. Here and in the following we assume that the matrix (a^{ij}) is real and symmetric.

The characterization of differential equation (3.1) follows from the signs of the eigenvalues of $(a^{ij}(x))$.

Definition. Differential equation (3.1) is said to be of *type* (α, β, γ) *at* $x \in \Omega$ if α eigenvalues of $(a^{ij})(x)$ are positive, β eigenvalues are negative and γ eigenvalues are zero $(\alpha + \beta + \gamma = n)$.
In particular, equation is called
elliptic if it is of type $(n, 0, 0)$ or of type $(0, n, 0)$, i. e., all eigenvalues are different from zero and have the same sign,
parabolic if it is of type $(n-1, 0, 1)$ or of type $(0, n-1, 1)$, i. e., one eigenvalue is zero and all the others are different from zero and have the same sign,
hyperbolic if it is of type $(n-1, 1, 0)$ or of type $(1, n-1, 0)$, i. e., all eigenvalues are different from zero and one eigenvalue has another sign than all the others.

Remarks:

1. According to this definition there are other types aside from elliptic, parabolic or hyperbolic equations.

2. The classification depends in general on $x \in \Omega$. An example is the Tricomi equation, which appears in the theory of transsonic flows,

$$yu_{xx} + u_{yy} = 0.$$

This equation is elliptic if $y > 0$, parabolic if $y = 0$ and hyperbolic for $y < 0$.

Examples:

1. The *Laplace equation* in $\mathbb{R}^3$ is $\triangle u = 0$, where

$$\triangle u := u_{xx} + u_{yy} + u_{zz}.$$

This equation is elliptic. Thus for each manifold $\mathcal{S}$ given by $\{(x, y, z) : \chi(x, y, z) = 0\}$, where χ is an arbitrary sufficiently regular function such that $\nabla\chi \neq 0$, all derivatives of u are known on $\mathcal{S}$, provided u and ∇u are known on $\mathcal{S}$.

2. The *wave equation* $u_{tt} = u_{xx} + u_{yy} + u_{zz}$, where $u = u(x, y, z, t)$, is hyperbolic. Such a type describes oscillations of mechanical structures, for example.

3. The *heat equation* $u_t = u_{xx}+u_{yy}+u_{zz}$, where $u = u(x, y, z, t)$, is parabolic. It describes, for example, the propagation of heat in a domain.

4. Consider the case that the (real) coefficients a^{ij} in equation (3.1) are *constant*. We recall that the matrix $A = (a^{ij})$ is symmetric, i. e., $A^T = A$. In this case, the transform to principle axis leads to a normal form from which the classification of the equation is obviously. Let U be the associated orthogonal matrix, then

$$U^TAU = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \multicolumn{4}{c}{\dotfill} \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

Here is $U = (z_1, \ldots, z_n)$, where z_l, $l = 1, \ldots, n$, is an orthonormal system of eigenvectors to the eigenvalues λ_l.

Set $y = U^T x$ and $v(y) = u(Uy)$, then

$$\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} = \sum_{i=1}^{n} \lambda_i v_{y_i y_j}. \tag{3.4}$$

3.1.1 Normal form in two variables

Consider the differential equation

$$a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} + \text{terms of lower order} = 0 \tag{3.5}$$

in $\Omega \subset \mathbb{R}^2$. The associated characteristic differential equation is

$$a\chi_x^2 + 2b\chi_x\chi_y + c\chi_y^2 = 0. \tag{3.6}$$

We show that an appropriate coordinate transform will simplify equation (3.5) sometimes in such a way that we can solve the transformed equation explicitly.

Let $z = \phi(x,y)$ be a solution of (3.6). Consider the level sets $\{(x,y) : \phi(x,y) = const.\}$ and assume $\phi_y \neq 0$ at a point (x_0, y_0) of the level set. Then there is a function $y(x)$ defined in a neighbourhood of x_0 such that $\phi(x, y(x)) = const$. It follows

$$y'(x) = -\frac{\phi_x}{\phi_y},$$

which implies, see the characteristic equation (3.6),

$$ay'^2 - 2by' + c = 0. \tag{3.7}$$

Then, provided $a \neq 0$, we can calculate $\mu := y'$ from the (known) coefficients a, b and c:

$$\mu_{1,2} = \frac{1}{a}\left(b \pm \sqrt{b^2 - ac}\right). \tag{3.8}$$

These solutions are real if and only of $ac - b^2 \leq 0$.

Equation (3.5) is hyperbolic if $ac - b^2 < 0$, parabolic if $ac - b^2 = 0$ and elliptic if $ac - b^2 > 0$. This follows from an easy discussion of the eigenvalues of the matrix

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix},$$

see an exercise.

Normal form of a hyperbolic equation

Let ϕ and ψ are solutions of the characteristic equation (3.6) such that

$$\begin{aligned} y_1' \equiv \mu_1 &= -\frac{\phi_x}{\phi_y} \\ y_2' \equiv \mu_2 &= -\frac{\psi_x}{\psi_y}, \end{aligned}$$

where μ_1 and μ_2 are given by (3.8). Thus ϕ and ψ are solutions of the linear homogeneous equations of first order

$$\phi_x + \mu_1(x,y)\phi_y = 0 \tag{3.9}$$

$$\psi_x + \mu_2(x,y)\psi_y = 0. \tag{3.10}$$

Assume $\phi(x,y)$, $\psi(x,y)$ are solutions such that $\nabla\phi \neq 0$ and $\nabla\psi \neq 0$, see an exercise for the existence of such solutions.

Consider two families of level sets defined by $\phi(x,y) = \alpha$ and $\psi(x,y) = \beta$, see Figure 3.3.

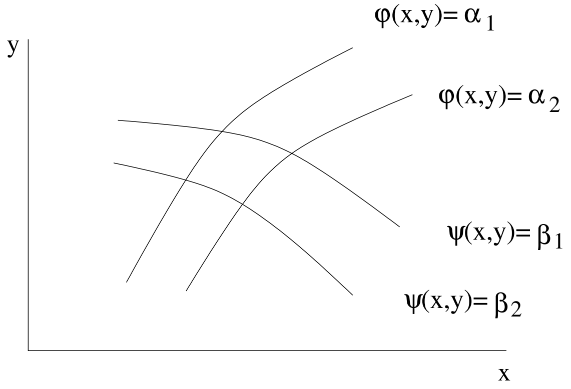

Figure 3.3: Level sets

These level sets are characteristic curves of the partial differential equations (3.9) and (3.10), respectively, see an exercise of the previous chapter.

Lemma. (i) *Curves from different families can not touch each other.*

(ii) $\phi_x\psi_y - \phi_y\psi_x \neq 0$.

Proof. (i):

$$y_2' - y_1' \equiv \mu_2 - \mu_1 = -\frac{2}{a}\sqrt{b^2 - ac} \neq 0.$$

(ii):

$$\mu_2 - \mu_1 = \frac{\phi_x}{\phi_y} - \frac{\psi_x}{\psi_y}.$$

□

Proposition 3.1. *The mapping* $\xi = \phi(x,y)$, $\eta = \psi(x,y)$ *transforms equation (3.5) into*

$$v_{\xi\eta} = \textit{lower order terms}, \tag{3.11}$$

where $v(\xi,\eta) = u(x(\xi,\eta), y(\xi,\eta))$.

Proof. The proof follows from a straightforward calculation.

$$\begin{aligned}
u_x &= v_\xi \phi_x + v_\eta \psi_x \\
u_y &= v_\xi \phi_y + v_\eta \psi_y \\
u_{xx} &= v_{\xi\xi}\phi_x^2 + 2v_{\xi\eta}\phi_x\psi_x + v_{\eta\eta}\psi_x^2 + \text{lower order terms} \\
u_{xy} &= v_{\xi\xi}\phi_x\phi_y + v_{\xi\eta}(\phi_x\psi_y + \phi_y\psi_x) + v_{\eta\eta}\psi_x\psi_y + \text{lower order terms} \\
u_{yy} &= v_{\xi\xi}\phi_y^2 + 2v_{\xi\eta}\phi_y\psi_y + v_{\eta\eta}\psi_y^2 + \text{lower order terms.}
\end{aligned}$$

Thus

$$au_{xx} + 2bu_{xy} + cu_{yy} = \alpha v_{\xi\xi} + 2\beta v_{\xi\eta} + \gamma v_{\eta\eta} + l.o.t.,$$

where

$$\begin{aligned}
\alpha : &= a\phi_x^2 + 2b\phi_x\phi_y + c\phi_y^2 \\
\beta : &= a\phi_x\psi_x + b(\phi_x\psi_y + \phi_y\psi_x) + c\phi_y\psi_y \\
\gamma : &= a\psi_x^2 + 2b\psi_x\psi_y + c\psi_y^2.
\end{aligned}$$

The coefficients α and γ are zero since ϕ and ψ are solutions of the characteristic equation. Since

$$\alpha\gamma - \beta^2 = (ac - b^2)(\phi_x\psi_y - \phi_y\psi_x)^2,$$

it follows from the above lemma that the coefficient β is different from zero.
□

Example: Consider the differential equation

$$u_{xx} - u_{yy} = 0.$$

The associated characteristic differential equation is

$$\chi_x^2 - \chi_y^2 = 0.$$

Since $\mu_1 = -1$ and $\mu_2 = 1$, the functions ϕ and ψ satisfy differential equations

$$\begin{aligned} \phi_x + \phi_y &= 0 \\ \psi_x - \psi_y &= 0. \end{aligned}$$

Solutions with $\nabla\phi \neq 0$ and $\nabla\psi \neq 0$ are

$$\phi = x - y, \quad \psi = x + y.$$

Thus the mapping

$$\xi = x - y, \quad \eta = x + y$$

leads to the simple equation

$$v_{\xi\eta}(\xi, \eta) = 0.$$

Assume $v \in C^2$ is a solution, then $v_\xi = f_1(\xi)$ for an arbitrary C^1 function $f_1(\xi)$. It follows

$$v(\xi, \eta) = \int_0^\xi f_1(\alpha)\ d\alpha + g(\eta),$$

where g is an arbitrary C^2 function. Thus each C^2-solution of the differential equation can be written as

$$(\star) \qquad v(\xi, \eta) = f(\xi) + g(\eta),$$

where $f,\ g \in C^2$. On the other hand, for arbitrary C^2-functions f, g the function $(\star)$ is a solution of the differential equation $v_{\xi\eta} = 0$. Consequently each C^2-solution of the original equation $u_{xx} - u_{yy} = 0$ is given by

$$u(x, y) = f(x - y) + g(x + y),$$

where $f,\ g \in C^2$.

3.2 Quasilinear equations of second order

Here we consider the equation

$$\sum_{i,j=1}^{n} a^{ij}(x,u,\nabla u)u_{x_ix_j}+b(x,u,\nabla u)=0 \tag{3.12}$$

in a domain $\Omega\subset\mathbb{R}^n$, where $u:\ \Omega\mapsto\mathbb{R}$. We assume that $a^{ij}=a^{ji}$.

As in the previous section we can derive the characteristic equation

$$\sum_{i,j=1}^{n} a^{ij}(x,u,\nabla u)\chi_{x_i}\chi_{x_j}=0.$$

In contrast to linear equations, solutions of the characteristic equation depend on the solution considered.

3.2.1 Quasilinear elliptic equations

There is a large class of quasilinear equations such that the associated characteristic equation has no solution χ, $\nabla\chi\neq 0$.

Set

$$U=\{(x,z,p):\ x\in\Omega,\ z\in\mathbb{R},\ p\in\mathbb{R}^n\}.$$

Definition. The quasilinear equation (3.12) is called *elliptic* if the matrix $(a^{ij}(x,z,p))$ is positive definite for each $(x,z,p)\in U$.

Assume equation (3.12) is elliptic and let $\lambda(x,z,p)$ be the minimum and $\Lambda(x,z,p)$ the maximum of the eigenvalues of (a^{ij}), then

$$0<\lambda(x,z,p)|\zeta|^2\le\sum_{i,j=1}^{n} a^{ij}(x,z,p)\zeta_i\zeta_j\le\Lambda(x,z,p)|\zeta|^2$$

for all $\zeta\in\mathbb{R}^n$.

Definition. Equation (3.12) is called *uniformly elliptic* if Λ/λ is uniformly bounded in U.

An important class of elliptic equations which are not uniformly elliptic (nonuniformly elliptic) is

$$\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\frac{u_{x_i}}{\sqrt{1+|\nabla u|^2}}\right)+\text{lower order terms}=0. \tag{3.13}$$

The main part is the minimal surface operator (left hand side of the minimal surface equation). The coefficients a^{ij} are

$$a^{ij}(x,z,p) = \left(1+|p|^2\right)^{-1/2}\left(\delta_{ij} - \frac{p_i p_j}{1+|p|^2}\right),$$

δ_{ij} denotes the Kronecker delta symbol. It follows that

$$\lambda = \frac{1}{(1+|p|^2)^{3/2}}, \quad \Lambda = \frac{1}{(1+|p|^2)^{1/2}}.$$

Thus equation (3.13) is not uniformly elliptic.

The behaviour of solutions of uniformly elliptic equations is similar to linear elliptic equations in contrast to the behaviour of solutions of nonuniformly elliptic equations. Typical examples for nonuniformly elliptic equations are the minimal surface equation and the capillary equation.

3.3 Systems of first order

Consider the quasilinear system

$$\sum_{k=1}^{n} A^k(x,u)u_{u_k} + b(x,u) = 0, \tag{3.14}$$

where A^k are $m \times m$-matrices, sufficiently regular with respect to their arguments, and

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}, \quad u_{x_k} = \begin{pmatrix} u_{1,x_k} \\ \vdots \\ u_{m,x_k} \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}.$$

We ask the same question as above: can we calculate all derivatives of u in a neighbourhood of a given hypersurface $\mathcal{S}$ in $\mathbb{R}^n$ defined by $\chi(x) = 0$, $\nabla\chi \neq 0$, provided $u(x)$ is given on $\mathcal{S}$?

For an answer we map $\mathcal{S}$ onto a flat surface $\mathcal{S}_0$ by using the mapping $\lambda = \lambda(x)$ of Section 3.1 and write equation (3.14) in new coordinates. Set $v(\lambda) = u(x(\lambda))$, then

$$\sum_{k=1}^{n} A^k(x,u)\chi_{x_k} v_{\lambda_n} = \text{terms known on } \mathcal{S}_0.$$

We can solve this system with respect to v_{λ_n}, provided that

$$\det\left(\sum_{k=1}^n A^k(x,u)\chi_{x_k}\right)\neq 0$$

on $\mathcal{S}$.

Definition. Equation

$$\det\left(\sum_{k=1}^n A^k(x,u)\chi_{x_k}\right)= 0$$

is called *characteristic equation* associated to equation (3.14) and a surface $\mathcal{S}$: $\chi(x)=0$, defined by a solution χ, $\nabla\chi\neq 0$, of this characteristic equation is said to be *characteristic surface*.

Set

$$C(x,u,\zeta)=\det\left(\sum_{k=1}^n A^k(x,u)\zeta_k\right)$$

for $\zeta\in\mathbb{R}^n$.

Definition. (i) The system (3.14) is *hyperbolic* at $(x,u(x))$ if there is a regular linear mapping $\zeta=Q\eta$, where $\eta=(\eta_1,\ldots,\eta_{n-1},\kappa)$, such that there exists m *real* roots $\kappa_k=\kappa_k(x,u(x),\eta_1,\ldots,\eta_{n-1})$, $k=1,\ldots,m$, of

$$D(x,u(x),\eta_1,\ldots,\eta_{n-1},\kappa)=0$$

for all $(\eta_1,\ldots,\eta_{n-1})$, where

$$D(x,u(x),\eta_1,\ldots,\eta_{n-1},\kappa)=C(x,u(x),x,Q\eta).$$

(ii) System (3.14) is *parabolic* if there exists a regular linear mapping $\zeta=Q\eta$ such that D is independent of κ, i. e., D depends on less than n parameters.

(iii) System (3.14) is *elliptic* if $C(x,u,\zeta)=0$ only if $\zeta=0$.

Remark. In the elliptic case all derivatives of the solution can be calculated from the given data and the given equation.

3.3.1 Examples

1. Beltrami equations

$$\begin{aligned} Wu_x - bv_x - cv_y &= 0 \qquad (3.15)\\ Wu_y + av_x + bv_y &= 0, \qquad (3.16)\end{aligned}$$

where $W,\ a,\ b,\ c$ are given functions depending of (x, y), $W \neq 0$ and the matrix

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

is positive definite.

The Beltrami system is a generalization of Cauchy-Riemann equations. The function $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$, is called a *quasiconform mapping*, see for example [9], Chapter 12, for an application to partial differential equations.

Set

$$A^1 = \begin{pmatrix} W & -b \\ 0 & a \end{pmatrix},\quad A^2 = \begin{pmatrix} 0 & -c \\ W & b \end{pmatrix}.$$

Then the system (3.15), (3.16) can be written as

$$A^1 \begin{pmatrix} u_x \\ v_x \end{pmatrix} + A^2 \begin{pmatrix} u_y \\ v_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus,

$$C(x, y, \zeta) = \begin{vmatrix} W\zeta_1 & -b\zeta_1 - c\zeta_2 \\ W\zeta_2 & a\zeta_1 + b\zeta_2 \end{vmatrix} = W(a\zeta_1^2 + 2b\zeta_1\zeta_2 + c\zeta_2^2),$$

which is different from zero if $\zeta \neq 0$ according to the above assumptions. Thus the *Beltrami system is elliptic.*

2. Maxwell equations

The Maxwell equations in the isotropic case are

$$\begin{aligned} c\ \mathrm{rot}_x\ H &= \lambda E + \epsilon E_t \qquad (3.17)\\ c\ \mathrm{rot}_x\ E &= -\mu H_t, \qquad (3.18)\end{aligned}$$

where
$E=(e_1,e_2,e_3)^T$ electric field strength, $e_i=e_i(x,t)$, $x=(x_1,x_2,x_3)$,
$H=(h_1,h_2,h_3)^T$ magnetic field strength, $h_i=h_i(x,t)$,
c speed of light,
λ specific conductivity,
ϵ dielectricity constant,
μ magnetic permeability.
Here $c,\ \lambda,\ \epsilon$ and μ are positive constants.

Set $p_0=\chi_t,\ p_i=\chi_{x_i},\ i=1,\dots 3$, then the characteristic differential equation is

$$\begin{vmatrix} \epsilon p_0/c & 0 & 0 & 0 & p_3 & -p_2 \\ 0 & \epsilon p_0/c & 0 & -p_3 & 0 & p_1 \\ 0 & 0 & \epsilon p_0/c & p_2 & -p_1 & 0 \\ 0 & -p_3 & p_2 & \mu p_0/c & 0 & 0 \\ p_3 & 0 & -p_1 & 0 & \mu p_0/c & 0 \\ -p_2 & p_1 & 0 & 0 & 0 & \mu p_0/c \end{vmatrix}=0.$$

The following manipulations simplifies this equation:
(i) multiply the first three columns with $\mu p_0/c$,
(ii) multiply the 5th column with $-p_3$ and the the 6th column with p_2 and add the sum to the 1st column,
(iii) multiply the 4th column with p_3 and the 6th column with $-p_1$ and add the sum to the 2th column,
(iv) multiply the 4th column with $-p_2$ and the 5th column with p_1 and add the sum to the 3th column,
(v) expand the resulting determinant with respect to the elements of the 6th, 5th and 4th row.

We obtain

$$\begin{vmatrix} q+p_1^2 & p_1p_2 & p_1p_3 \\ p_1p_2 & q+p_2^2 & p_2p_3 \\ p_1p_3 & p_2p_3 & q+p_3^2 \end{vmatrix}=0,$$

where

$$q:=\frac{\epsilon\mu}{c^2}p_0^2-g^2$$

with $g^2:=p_1^2+p_2^2+p_3^2$. The evaluation of the above equation leads to $q^2(q+g^2)=0$, i. e.,

$$\chi_t^2\left(\frac{\epsilon\mu}{c^2}\chi_t^2-|\nabla_x\chi|^2\right)=0.$$

It follows immediately that *Maxwell equations* are a *hyperbolic system*, see an exercise. There are two solutions of this characteristic equation. The first one are characteristic surfaces $\mathcal{S}(t)$, defined by $\chi(x,t) = 0$, which satisfy $\chi_t = 0$. These surfaces are called *stationary waves*. The second type of characteristic surfaces are defined by solutions of

$$\frac{\epsilon\mu}{c^2}\chi_t^2 = |\nabla_x \chi|^2.$$

Functions defined by $\chi = f(n\cdot x - Vt)$ are solutions of this equation. Here is $f(s)$ an arbitrary function with $f'(s) \neq 0$, n is a unit vector and $V = c/\sqrt{\epsilon\mu}$. The associated characteristic surfaces $\mathcal{S}(t)$ are defined by

$$\chi(x,t) \equiv f(n \cdot x - Vt) = 0,$$

here we assume that 0 is in he range of $f:\ \mathbb{R} \mapsto \mathbb{R}$. Thus, $\mathcal{S}(t)$ is defined by $n \cdot x - Vt = c$, where c is a fixed constant. It follows that the planes $\mathcal{S}(t)$ with normal n move with speed V in direction of n, see Figure 3.4

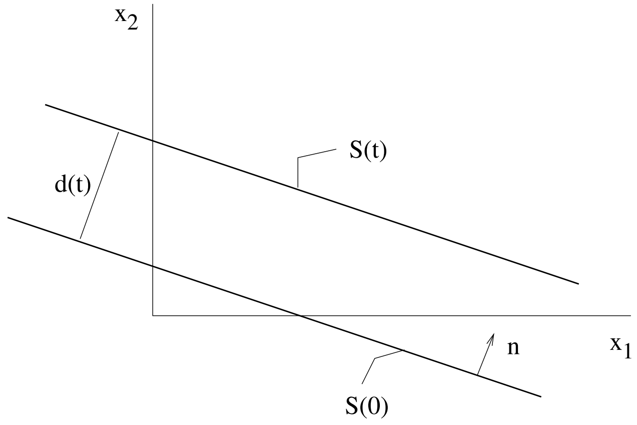

Figure 3.4: $d'(t)$ is the speed of plane waves

V is called *speed* of the plane wave $\mathcal{S}(t)$.

Remark. According to the previous discussions, singularities of a solution of Maxwell equations are located at most on characteristic surfaces.

A special case of Maxwell equations are the **telegraph equations**, which follow from Maxwell equations if div $E = 0$ and div $H = 0$, i. e., E and

H are fields free of sources. In fact, it is sufficient to assume that this assumption is satisfied at a fixed time t_0 only, see an exercise.
Since

$$\mathrm{r}ot_x\ \mathrm{r}ot_x\ A = \mathrm{grad}_x\ \mathrm{d}iv_x\ A - \triangle_x A$$

for each C^2-vector field A, it follows from Maxwell equations the uncoupled system

$$\begin{aligned}\triangle_x E &= \frac{\epsilon\mu}{c^2} E_{tt} + \frac{\lambda\mu}{c^2} E_t \\ \triangle_x H &= \frac{\epsilon\mu}{c^2} H_{tt} + \frac{\lambda\mu}{c^2} H_t.\end{aligned}$$

3. Equations of gas dynamics

Consider the following quasilinear equations of first order.

$$v_t + (v \cdot \nabla_x)\ v + \frac{1}{\rho}\nabla_x p = f \quad \text{(Euler equations).}$$

Here is
$v = (v_1, v_2, v_3)$ the vector of speed, $v_i = v_i(x,t)$, $x = (x_1, x_2, x_3)$,
p pressure, $p = (x,t)$,
ρ density, $\rho = \rho(x,t)$,
$f = (f_1, f_2, f_3)$ density of the external force, $f_i = f_i(x,t)$,
$(v \cdot \nabla_x)v \equiv (v \cdot \nabla_x v_1, v \cdot \nabla_x v_2, v \cdot \nabla_x v_3))^T$.

The second equation is

$$\rho_t + v \cdot \nabla_x \rho + \rho\ \mathrm{d}iv_x\ v = 0 \quad \text{(conservation of mass).}$$

Assume the gas is compressible and that there is a function (state equation)

$$p = p(\rho),$$

where $p'(\rho) > 0$ if $\rho > 0$. Then the above system of four equations is

$$\begin{aligned} v_t + (v \cdot \nabla)v + \frac{1}{\rho}p'(\rho)\nabla\rho &= f & (3.19)\\ \rho_t + \rho\ \mathrm{d}iv\ v + v \cdot \nabla\rho &= 0, & (3.20)\end{aligned}$$

where $\nabla \equiv \nabla_x$ and $\mathrm{d}iv \equiv \mathrm{d}iv_x$, i. e., these operators apply on the spatial variables only.

The characteristic differential equation is here

$$\begin{vmatrix} \frac{d\chi}{dt} & 0 & 0 & \frac{1}{\rho}p'\chi_{x_1} \\ 0 & \frac{d\chi}{dt} & 0 & \frac{1}{\rho}p'\chi_{x_2} \\ 0 & 0 & \frac{d\chi}{dt} & \frac{1}{\rho}p'\chi_{x_3} \\ \rho\chi_{x_1} & \rho\chi_{x_2} & \rho\chi_{x_3} & \frac{d\chi}{dt} \end{vmatrix} = 0,$$

where

$$\frac{d\chi}{dt} := \chi_t + (\nabla_x\chi)\cdot v.$$

Evaluating the determinant, we get the characteristic differential equation

$$\left(\frac{d\chi}{dt}\right)^2\left(\left(\frac{d\chi}{dt}\right)^2 - p'(\rho)|\nabla_x\chi|^2\right) = 0. \tag{3.21}$$

This equation implies consequences for the speed of the characteristic surfaces as the following consideration shows.

Consider a family $\mathcal{S}(t)$ of surfaces in $\mathbb{R}^3$ defined by $\chi(x,t) = c$, where $x \in \mathbb{R}^3$ and c is a fixed constant. As usually, we assume that $\nabla_x\chi \neq 0$. One of the two normals on $\mathcal{S}(t)$ at a point of the surface $\mathcal{S}(t)$ is given by, see an exercise,

$$\mathbf{n} = \frac{\nabla_x\chi}{|\nabla_x\chi|}. \tag{3.22}$$

Let $Q_0 \in \mathcal{S}(t_0)$ and let $Q_1 \in \mathcal{S}(t_1)$ be a point on the line defined by $Q_0 + s\mathbf{n}$, where $\mathbf{n}$ is the normal (3.22 on $\mathcal{S}(t_0)$ at Q_0 and $t_0 < t_1$, $t_1 - t_0$ small, see Figure 3.5.

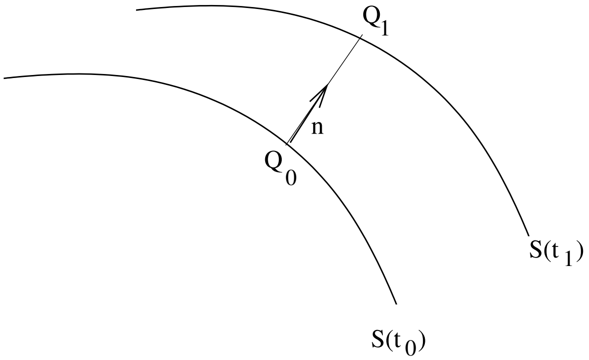

Figure 3.5: Definition of the speed of a surface

Definition. The limit

$$P = \lim_{t_1 \to t_0} \frac{|Q_1 - Q_0|}{t_1 - t_0}$$

is called *speed* of the surface $\mathcal{S}(t)$.

Proposition 3.2. *The speed of the surface* $\mathcal{S}(t)$ *is*

$$P = -\frac{\chi_t}{|\nabla_x \chi|}. \tag{3.23}$$

Proof. The proof follows from $\chi(Q_0, t_0) = 0$ and $\chi(Q_0 + d\mathbf{n}, t_0 + \triangle t) = 0$, where $d = |Q_1 - Q_0|$ and $\triangle t = t_1 - t_0$.

□

Set $v_n := v \cdot \mathbf{n}$ which is the component of the velocity vector in direction $\mathbf{n}$. From (3.22) we get

$$v_n = \frac{1}{|\nabla_x \chi|} v \cdot \nabla_x \chi.$$

Definition. $V := P - v_n$, the difference of the speed of the surface and the speed of liquid particles, is called *relative speed.*

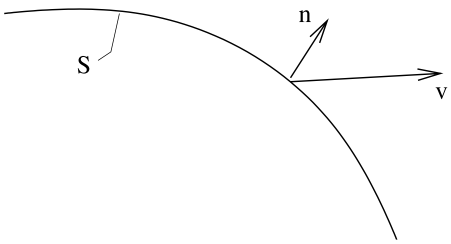

Figure 3.6: Definition of relative speed

Using the above formulas for P and v_n it follows

$$V = P - v_n = -\frac{\chi_t}{|\nabla_x \chi|} - \frac{v \cdot \nabla_x \chi}{|\nabla_x \chi|} = -\frac{1}{|\nabla_x \chi|} \frac{d\chi}{dt}.$$

Then, we obtain from the characteristic equation (3.21) that

$$V^2 |\nabla_x \chi|^2 \left(V^2 |\nabla_x \chi|^2 - p'(\rho) |\nabla_x \chi|^2 \right) = 0.$$

An interesting conclusion is that there are two relative speeds: $V = 0$ or $V^2 = p'(\rho)$.

Definition. $\sqrt{p'(\rho)}$ is called *speed of sound* .

3.4 Systems of second order

Here we consider the system

$$\sum_{k,l=1}^{n} A^{kl}(x, u, \nabla u) u_{x_k x_l} + \text{lower order terms} = 0, \tag{3.24}$$

where A^{kl} are $(m \times m)$ matrices and $u = (u_1, \ldots, u_m)^T$. We assume $A^{kl} = A^{lk}$, which is no restriction of generality provided $u \in C^2$ is satisfied. As in the previous sections, the classification follows from the question whether or not we can calculate formally the solution from the differential equations, if sufficiently many data are given on an initial manifold. Let the initial manifold $\mathcal{S}$ be given by $\chi(x) = 0$ and assume that $\nabla\chi \neq 0$. The mapping $x = x(\lambda)$, see previous sections, leads to

$$\sum_{k,l=1}^{n} A^{kl} \chi_{x_k} \chi_{x_l} v_{\lambda_n \lambda_n} = \text{terms known on } \mathcal{S},$$

where $v(\lambda) = u(x(\lambda))$.

The characteristic equation is here

$$\det \left(\sum_{k,l=1}^{n} A^{kl} \chi_{x_k} \chi_{x_l} \right) = 0.$$

If there is a solution χ with $\nabla\chi \neq 0$, then it is possible that second derivatives are not continuous in a neighbourhood of $\mathcal{S}$.

Definition. The system is called *elliptic* if

$$\det \left(\sum_{k,l=1}^{n} A^{kl} \zeta_k \zeta_l \right) \neq 0$$

for all $\zeta \in \mathbb{R}^n$, $\zeta \neq 0$.

3.4.1 Examples

1. Navier-Stokes equations

The Navier-Stokes system for a viscous incompressible liquid is

$$\begin{aligned} v_t + (v \cdot \nabla_x)v &= -\frac{1}{\rho}\nabla_x p + \gamma \triangle_x v \\ \mathrm{d}iv_x\ v &= 0, \end{aligned}$$

where ρ is the (constant and positive) density of liquid,
γ is the (constant and positive) viscosity of liquid,
$v = v(x,t)$ velocity vector of liquid particles, $x \in \mathbb{R}^3$ or in $\mathbb{R}^2$,
$p = p(x,t)$ pressure.

The problem is to find solutions $v,\ p$ of the above system.

2. Linear elasticity

Consider the system

$$\rho \frac{\partial^2 u}{\partial t^2} = \mu \triangle_x u + (\lambda + \mu)\nabla_x(\mathrm{d}iv_x\ u) + f. \tag{3.25}$$

Here is, in the case of an elastic body in $\mathbb{R}^3$,
$u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ displacement vector,
$f(x,t)$ density of external force,
ρ (constant) density,
$\lambda,\ \mu$ (positive) Lamé constants.

The characteristic equation is $\det C = 0$, where the entries of the matrix C are given by

$$c_{ij} = (\lambda + \mu)\chi_{x_i}\chi_{x_j} + \delta_{ij}\left(\mu|\nabla_x \chi|^2 - \rho\chi_t^2\right).$$

The characteristic equation is

$$\left((\lambda + 2\mu)|\nabla_x\chi|^2 - \rho\chi_t^2\right)\left(\mu|\nabla_x\chi|^2 - \rho\chi_t^2\right)^2 = 0.$$

It follows that two different speeds P of characteristic surfaces $\mathcal{S}(t)$, defined by $\chi(x,t) = const.$, are possible, namely

$$P_1 = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad \text{and} \quad P_2 = \sqrt{\frac{\mu}{\rho}}.$$

We recall that $P = -\chi_t/|\nabla_x\chi|$.

3.5 Theorem of Cauchy-Kovalevskaya

Consider the quasilinear system of first order (3.14) of Section 3.3. Assume an initial manifolds $\mathcal{S}$ is given by $\chi(x) = 0$, $\nabla\chi \neq 0$, and suppose that χ is not characteristic. Then, see Section 3.3, the system (3.14) can be written as

$$\begin{aligned} u_{x_n} &= \sum_{i=1}^{n-1} a^i(x,u)u_{x_i} + b(x,u) && (3.26)\\ u(x_1,\ldots,x_{n-1},0) &= f(x_1,\ldots,x_{n-1}) && (3.27) \end{aligned}$$

Here is $u = (u_1,\ldots,u_m)^T$, $b = (b_1,\ldots,b_n)^T$ and a^i are $(m\times m)$-matrices. We assume a^i, b and f are in C^∞ with respect to their arguments. From (3.26) and (3.27) it follows that we can calculate formally all derivatives $D^\alpha u$ in a neigbourhood of the plane $\{x:\ x_n = 0\}$, in particular in a neighbourhood of $0 \in \mathbb{R}^n$. Thus we have a formal power series of $u(x)$ at $x = 0$:

$$u(x) \sim \sum \frac{1}{\alpha!} D^\alpha u(0) x^\alpha.$$

For notations and definitions used here and in the following see the appendix to this section.

Then, as usually, two questions arise:

(i) Does the power series converge in a neighbourhood of $0 \in \mathbb{R}^n$?

(ii) Is a convergent power series a solution of the initial value problem (3.26), (3.27)?

Remark. Quite different to this power series method is the method of *asymptotic expansions*. Here one is interested in a good approximation of an unknown solution of an equation by a finite sum $\sum_{i=0}^N \phi_i(x)$ of functions ϕ_i. In general, the infinite sum $\sum_{i=0}^\infty \phi_i(x)$ does not converge, in contrast to the power series method of this section. See [15] for some asymptotic formulas in capillarity.

Theorem 3.1 (Cauchy-Kovalevskaya). *There is a neighbourhood of* $0 \in \mathbb{R}^n$ *such there is a real analytic solution of the initial value problem (3.26), (3.27). This solution is unique in the class of real analytic functions.*

Proof. The proof is taken from F. John [10]. We introduce $u - f$ as the new solution for which we are looking at and we add a new coordinate $u^\star$ to the solution vector by setting $u^\star(x) = x_n$. Then

$$u^\star_{x_n} = 1,\ u^\star_{x_k} = 0,\ k = 1, \ldots, n-1,\ u^\star(x_1, \ldots, x_{n-1}, 0) = 0$$

and the extended system (3.26), (3.27) is

$$\begin{pmatrix} u_{1,x_n} \\ \vdots \\ u_{m,x_n} \\ u^\star_{x_n} \end{pmatrix} = \sum_{i=1}^{n-1} \begin{pmatrix} a^i & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_{1,x_i} \\ \vdots \\ u_{m,x_i} \\ u^\star_{x_i} \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_m \\ 1 \end{pmatrix},$$

where the associated initial condition is $u(x_1, \ldots, x_{n-1}, 0) = 0$. The new u is $u = (u_1, \ldots, u_m)^T$, the new a^i are $a^i(x_1, \ldots, x_{n-1}, u_1, \ldots, u_m, u^\star)$ and the new b is $b = (x_1, \ldots, x_{n-1}, u_1, \ldots, u_m, u^\star)^T$.

Thus we are led to an initial value problem of the type

$$\begin{aligned} u_{j,x_n} &= \sum_{i=1}^{n-1} \sum_{k=1}^{N} a^i_{jk}(z) u_{k,x_i} + b_j(z),\ j = 1, \ldots, N && (3.28) \\ u_j(x) &= 0 \text{ if } x_n = 0, && (3.29) \end{aligned}$$

where $j = 1, \ldots, N$ and $z = (x_1, \ldots, x_{n-1}, u_1, \ldots, u_N)$.

The point here is that a^i_{jk} and b_j are independent of x_n. This fact simplifies the proof of the theorem.

From (3.28) and (3.29) we can calculate formally all $D^\beta u_j$. Then we have formal power series for u_j:

$$u_j(x) \sim \sum_\alpha c^{(j)}_\alpha x^\alpha,$$

where

$$c^{(j)}_\alpha = \frac{1}{\alpha!} D^\alpha u_j(0).$$

We will show that these power series are (absolutely) convergent in a neighbourhood of $0 \in \mathbb{R}^n$, i. e., they are real analytic functions, see the appendix for the definition of real analytic functions. Inserting these functions into the left and into the right hand side of (3.28) we obtain on the right and on the left hand side real analytic functions. This follows since compositions of real analytic functions are real analytic again, see Proposition A7 of the appendix to this section. The resulting power series on the left and on the right have the same coefficients caused by the calculation of the derivatives $D^\alpha u_j(0)$ from (3.28). It follows that $u_j(x)$, $j=1,\ldots,n$, defined by its formal power series are solutions of the initial value problem (3.28), (3.29).

Set

$$d=\left(\frac{\partial}{\partial z_1},\ldots,\frac{\partial}{\partial z_{N+n-1}}\right)$$

Lemma A. *Assume* $u\in C^\infty$ *in a neighbourhood of* $0\in\mathbb{R}^n$. *Then*

$$D^\alpha u_j(0)=P_\alpha\left(d^\beta a^i_{jk}(0),d^\gamma b_j(0)\right),$$

where $|\beta|,\ |\gamma|\le|\alpha|$ *and* P_α *are polynomials in the indicated arguments with* **nonnegative** *integers as coefficients which are* **independent** *of* a^i *and of* b.

Proof. It follows from equation (3.28) that

$$D_nD^\alpha u_j(0)=P_\alpha(d^\beta a^i_{jk}(0),d^\gamma b_j(0),D^\delta u_k(0)). \tag{3.30}$$

Here is $D_n=\partial/\partial x_n$ and $\alpha,\ \beta,\ \gamma,\ \delta$ satisfy the inequalities

$$|\beta|,\ |\gamma|\le|\alpha|,\quad |\delta|\le|\alpha|+1,$$

and, which is essential in the proof, the last coordinates in the multi-indices $\alpha=(\alpha_1,\ldots,\alpha_n)$, $\delta=(\delta_1,\ldots,\delta_n)$ satisfy $\delta_n\le\alpha_n$ since the right hand side of (3.28) is independent of x_n. Moreover, it follows from (3.28) that the polynomials P_α have integers as coefficients. The initial condition (3.29) implies

$$D^\alpha u_j(0)=0, \tag{3.31}$$

where $\alpha=(\alpha_1,\ldots,\alpha_{n-1},0)$, that is, $\alpha_n=0$. Then, the proof is by induction with respect to α_n. The induction starts with $\alpha_n=0$, then we replace $D^\delta u_k(0)$ in the right hand side of (3.30) by (3.31), that is by zero. Then it follows from (3.30) that

$$D^\alpha u_j(0)=P_\alpha(d^\beta a^i_{jk}(0),d^\gamma b_j(0),D^\delta u_k(0)),$$

where $\alpha=(\alpha_1,\ldots,\alpha_{n-1},1)$. □

Definition. Let $f=(f_1,\ldots,f_m)$, $F=(F_1,\ldots,F_m)$, $f_i=f_i(x)$, $F_i=F_i(x)$, and $f,\ F\in C^\infty$. We say f is *majorized* by F if

$$|D^\alpha f_k(0)|\le D^\alpha F_k(0),\quad k=1,\ldots,m$$

for all α. We write $f << F$, if f is majorized by F.

Definition. The initial value problem

$$\begin{aligned} U_{j,x_n} &= \sum_{i=1}^{n-1}\sum_{k=1}^{N} A^i_{jk}(z)U_{k,x_i} + B_j(z) && (3.32)\\ U_j(x) &= 0 \qquad \text{if } x_n = 0, && (3.33)\end{aligned}$$

$j = 1,\ldots,N$, A^i_{jk}, B_j real analytic, ist called *majorizing problem* to (3.28), (3.29) if

$$a^i_{jk} << A^i_{jk} \;\text{ and } b_j << B_j.$$

Lemma B. *The formal power series*

$$\sum_\alpha \frac{1}{\alpha!}D^\alpha u_j(0)x^\alpha,$$

where $D^\alpha u_j(0)$ are defined in Lemma A, is convergent in a neighbourhood of $0 \in \mathbb{R}^n$ if there exists a majorizing problem which has a real analytic solution U in $x = 0$, and

$$|D^\alpha u_j(0)| \le D^\alpha U_j(0).$$

Proof. It follows from Lemma A and from the assumption of Lemma B that

$$\begin{aligned} |D^\alpha u_j(0)| &\le P_\alpha\left(|d^\beta a^i_{jk}(0)|, |d^\gamma b_j(0)|\right)\\ &\le P_\alpha\left(|d^\beta A^i_{jk}(0)|, |d^\gamma B_j(0)|\right) \equiv D^\alpha U_j(0).\end{aligned}$$

The formal power series

$$\sum_\alpha \frac{1}{\alpha!}D^\alpha u_j(0)x^\alpha,$$

is convergent since

$$\sum_\alpha \frac{1}{\alpha!}|D^\alpha u_j(0)x^\alpha| \le \sum_\alpha \frac{1}{\alpha!}D^\alpha U_j(0)|x^\alpha|.$$

The right hand side is convergent in a neighbourhood of $x \in \mathbb{R}^n$ by assumption. □

Lemma C. *There is a majorising problem which has a real analytic solution.*

Proof. Since $a^i_{ij}(z)$, $b_j(z)$ are real analytic in a neighbourhood of $z=0$ it follows from Proposition A5 of the appendix to this section that there are positive constants M and r such that all these functions are majorized by

$$\frac{Mr}{r-z_1-\ldots-z_{N+n-1}}.$$

Thus a majorizing problem is

$$\begin{aligned} U_{j,x_n} &= \frac{Mr}{r-x_1-\ldots-x_{n-1}-U_1-\ldots-U_N}\left(1+\sum_{i=1}^{n-1}\sum_{k=1}^{N}U_{k,x_i}\right) \\ U_j(x) &= 0 \ \text{ if } x_n=0, \end{aligned}$$

$j=1,\ldots,N$.

The solution of this problem is

$$U_j(x_1,\ldots,x_{n-1},x_n)=V(x_1+\ldots+x_{n-1},x_n),\ j=1,\ldots,N,$$

where $V(s,t)$, $s=x_1+\ldots+x_{n-1}$, $t=x_n$, is the solution of the Cauchy initial value problem

$$\begin{aligned} V_t &= \frac{Mr}{r-s-NV}\left(1+N(n-1)V_s\right), \\ V(s,0) &= 0. \end{aligned}$$

which has the solution, see an exercise,

$$V(s,t)=\frac{1}{Nn}\left(r-s-\sqrt{(r-s)^2-2nMNrt}\right).$$

This function is real analytic in (s,t) at $(0,0)$. It follows that $U_j(x)$ are also real analytic functions. Thus the Cauchy-Kovalevskaya theorem is shown. □

Examples:

1. Ordinary differential equations

Consider the initial value problem

$$\begin{aligned} y'(x) &= f(x, y(x)) \\ y(x_0) &= y_0, \end{aligned}$$

where $x_0 \in \mathbb{R}$ and $y_0 \in \mathbb{R}^n$ are given. Assume $f(x, y)$ is real analytic in a neighbourhood of $(x_0, y_0) \in \mathbb{R} \times \mathbb{R}^n$. Then it follows from the above theorem that there exists an analytic solution $y(x)$ of the initial value problem in a neighbourhood of x_0. This solution is unique in the class of analytic functions according to the theorem of Cauchy-Kovalevskaya. From the Picard-Lindelöf theorem it follows that this analytic solution is unique even in the class of C^1-functions.

2. Partial differential equations of second order

Consider the boundary value problem for two variables

$$\begin{aligned} u_{yy} &= f(x, y, u, u_x, u_y, u_{xx}, u_{xy}) \\ u(x, 0) &= \phi(x) \\ u_y(x, 0) &= \psi(x). \end{aligned}$$

We assume that ϕ, ψ are analytic in a neighbourhood of $x = 0$ and that f is real analytic in a neighbourhood of

$$(0, 0, \phi(0), \phi'(0), \psi(0), \psi'(0)).$$

There exists a real analytic solution in a neigbourhood of $0 \in \mathbb{R}^2$ *of the above initial value problem.*

In particular, there is a real analytic solution in a neigbourhood of $0 \in \mathbb{R}^2$ of the initial value problem

$$\begin{aligned} \triangle u &= 1 \\ u(x, 0) &= 0 \\ u_y(x, 0) &= 0. \end{aligned}$$

The proof follows by writing the above problem as a system. Set $p = u_x$, $q = u_y$, $r = u_{xx}$, $s = u_{xy}$, $t = u_{yy}$, then

$$t = f(x, y, u, p, q, r, s).$$

Set $U = (u, p, q, r, s, t)^T$, $b = (q, 0, t, 0, 0, f_y + f_u q + f_q t)^T$ and

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & f_p & 0 & f_r & f_s \end{pmatrix}.$$

Then the rewritten differential equation is the system $U_y = AU_x + b$ with the initial condition

$$U(x, 0) = \left(\phi(x), \phi'(x), \psi(x), \phi''(x), \psi'(x), f_0(x)\right),$$

where $f_0(x) = f(x, 0, \phi(x), \phi'(x), \psi(x), \phi''(x), \psi'(x))$.

3.5.1 Appendix: Real analytic functions

Multi-index notation

The following multi-index notation simplifies many presentations of formulas. Let $x = (x_1, \ldots, x_n)$ and

$$u : \ \Omega \subset \mathbb{R}^n \mapsto \mathbb{R} \ \ (\text{or } \mathbb{R}^m \text{ for systems}).$$

The n-tuple of nonnegative integers (including zero)

$$\alpha = (\alpha_1, \ldots, \alpha_n)$$

is called *multi-index*. Set

$$\begin{aligned} |\alpha| &= \alpha_1 + \ldots + \alpha_n \\ \alpha! &= \alpha_1!\alpha_2! \cdot \ldots \cdot \alpha_n! \\ x^\alpha &= x_1^{\alpha_1} x_2^{\alpha_2} \cdot \ldots \cdot x_n^{\alpha_n} \quad \text{(for a monom)} \\ D_k &= \frac{\partial}{\partial x_k} \\ D &= (D_1, \ldots, D_n) \\ Du &= (D_1 u, \ldots, D_n u) \equiv \nabla u \equiv grad\ u \\ D^\alpha &= D_1^{\alpha_1} D_2^{\alpha_2} \cdot \ldots \cdot D_n^{\alpha_n} \equiv \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}. \end{aligned}$$

Define a partial order by

$$\alpha \geq \beta \text{ if and only if } \alpha_i \geq \beta_i \text{ for all } i.$$

Sometimes we use the notations

$$\mathbf{0} = (0, 0 \ldots, 0), \quad \mathbf{1} = (1, 1 \ldots, 1),$$

where $\mathbf{0},\ \mathbf{1} \in \mathbb{R}^n$.

Using this multi-index notion, we have

1.

$$(x+y)^\alpha = \sum_{\substack{\beta,\gamma \\ \beta+\gamma=\alpha}} \frac{\alpha!}{\beta!\gamma!} x^\beta y^\gamma,$$

where $x,\ y \in \mathbb{R}^n$ and $\alpha,\ \beta,\ \gamma$ are multi-indices.

2. Taylor expansion for a *polynomial* $f(x)$ of degree m:

$$f(x) = \sum_{|\alpha| \leq m} \frac{1}{\alpha!} \left(D^\alpha f(0)\right) x^\alpha,$$

here is $D^\alpha f(0) := \left(D^\alpha f(x)\right)|_{x=0}$.

3. Let $x = (x_1, \ldots, x_n)$ and $m \geq 0$ an integer, then

$$(x_1 + \ldots + x_n)^m = \sum_{|\alpha| = m} \frac{m!}{\alpha!} x^\alpha.$$

4.

$$\alpha! \leq |\alpha|! \leq n^{|\alpha|} \alpha!.$$

5. Leibniz's rule:

$$D^\alpha(fg) = \sum_{\substack{\beta,\gamma \\ \beta+\gamma=\alpha}} \frac{\alpha!}{\beta!\gamma!} (D^\beta f)(D^\gamma g).$$

6.

$$\begin{aligned} D^\beta x^\alpha &= \frac{\alpha!}{(\alpha-\beta)!} x^{\alpha-\beta} \ \text{ if } \alpha \geq \beta, \\ D^\beta x^\alpha &= 0 \ \text{ otherwise.} \end{aligned}$$

7. Directional derivative:

$$\frac{d^m}{dt^m} f(x+ty) = \sum_{|\alpha|=m} \frac{|\alpha|!}{\alpha!} \left(D^\alpha f(x+ty)\right) y^\alpha,$$

where $x,\ y \in \mathbb{R}^n$ and $t \in \mathbb{R}$.

8. Taylor's theorem: Let $u \in C^{m+1}$ in a neighbourhood $N(y)$ of y, then, if $x \in N(y)$,

$$u(x) = \sum_{|\alpha|\leq m} \frac{1}{\alpha!} \left(D^\alpha u(y)\right) (x-y)^\alpha + R_m,$$

where

$$R_m = \sum_{|\alpha|=m+1} \frac{1}{\alpha!} \left(D^\alpha u(y+\delta(x-y))\right) x^\alpha, \ 0 < \delta < 1,$$

$\delta = \delta(u, m, x, y)$, or

$$R_m = \frac{1}{m!} \int_0^1 (1-t)^m \Phi^{(m+1)}(t)\ dt,$$

where $\Phi(t) = u(y+t(x-y))$. It follows from **7.** that

$$R_m = (m+1) \sum_{|\alpha|=m+1} \frac{1}{\alpha!} \left(\int_0^1 (1-t) D^\alpha u(y+t(x-y))\ dt \right) (x-y)^\alpha.$$

9. Using multi-index notation, the general linear partial differential equation of order m can be written as

$$\sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha u = f(x) \ \text{ in } \Omega \subset \mathbb{R}^n.$$

Power series

Here we collect some definitions and results for power series in $\mathbb{R}^n$.

Definition. Let $c_\alpha \in \mathbb{R}$ (or $\in \mathbb{R}^m$). The series

$$\sum_\alpha c_\alpha \equiv \sum_{m=0}^\infty \left(\sum_{|\alpha|=m} c_\alpha \right)$$

is said to be convergent if

$$\sum_\alpha |c_\alpha| \equiv \sum_{m=0}^\infty \left(\sum_{|\alpha|=m} |c_\alpha| \right)$$

is convergent.

Remark. According to the above definition, a convergent series is absolutely convergent. It follows that we can rearrange the order of summation.

Using the above multi-index notation and keeping in mind that we can rearrange convergent series, we have

10. Let $x \in \mathbb{R}^n$, then

$$\begin{aligned} \sum_\alpha x^\alpha &= \prod_{i=1}^n \left(\sum_{\alpha_i=0}^\infty x_i^{\alpha_i} \right) \\ &= \frac{1}{(1-x_1)(1-x_2)\cdot \ldots \cdot (1-x_n)} \\ &= \frac{1}{(\mathbf{1}-x)^{\mathbf{1}}}, \end{aligned}$$

provided $|x_i| < 1$ is satisfied for each i.

11. Assume $x \in \mathbb{R}^n$ and $|x_1| + |x_2| + \ldots + |x_n| < 1$, then

$$\begin{aligned} \sum_\alpha \frac{|\alpha|!}{\alpha!} x^\alpha &= \sum_{j=0}^\infty \sum_{|\alpha|=j} \frac{|\alpha|!}{\alpha!} x^\alpha \\ &= \sum_{j=0}^\infty (x_1 + \ldots + x_n)^j \\ &= \frac{1}{1-(x_1+\ldots+x_n)}. \end{aligned}$$

12. Let $x \in \mathbb{R}^n$, $|x_i| < 1$ for all i, and β is a given multi-index. Then

$$\begin{aligned}\sum_{\alpha \ge \beta} \frac{\alpha!}{(\alpha - \beta)!} x^{\alpha - \beta} &= D^\beta \frac{1}{(\mathbf{1} - x)^{\mathbf{1}}} \\ &= \frac{\beta!}{(\mathbf{1} - x)^{\mathbf{1}+\beta}} \ .\end{aligned}$$

13. Let $x \in \mathbb{R}^n$ and $|x_1| + \ldots + |x_n| < 1$. Then

$$\begin{aligned}\sum_{\alpha \ge \beta} \frac{|\alpha|!}{(\alpha - \beta)!} x^{\alpha - \beta} &= D^\beta \frac{1}{1 - x_1 - \ldots - x_n} \\ &= \frac{|\beta|!}{(1 - x_1 - \ldots - x_n)^{1+|\beta|}} \ .\end{aligned}$$

Consider the power series

$$\sum_\alpha c_\alpha x^\alpha \tag{3.34}$$

and assume this series is convergent for a $z \in \mathbb{R}^n$. Then, by definition,

$$\mu := \sum_\alpha |c_\alpha||z^\alpha| < \infty$$

and the series (3.34) is uniformly convergent for all $x \in Q(z)$, where

$$Q(z): \quad |x_i| \le |z_i| \ \text{ for all } \ i.$$

Thus the power series (3.34) defines a continuous function defined on $Q(z)$, according to a theorem of Weierstrass.

The interior of $Q(z)$ is not empty if and only if $z_i \neq 0$ for all i, see Figure 3.7. For given x in a fixed compact subset D of $Q(z)$ there is a q, $0 < q < 1$, such that

$$|x_i| \le q|z_i| \ \text{ for all } i.$$

Set

$$f(x) = \sum_\alpha c_\alpha x^\alpha.$$

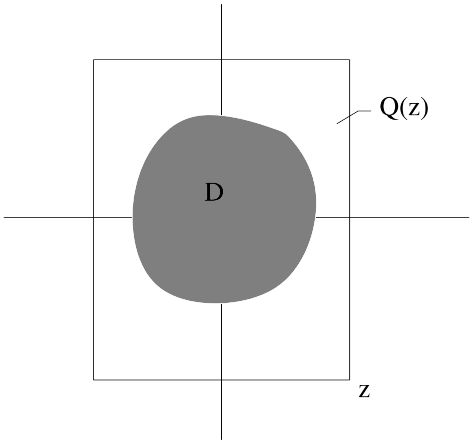

Figure 3.7: Definition of $D \in Q(z)$

Proposition A1. (i) *In every compact subset D of $Q(z)$ one has $f \in C^\infty(D)$ and the formal differentiate series, that is $\sum_\alpha D^\beta c_\alpha x^\alpha$, is uniformly convergent on the closure of D and is equal to $D^\beta f$.*

(ii)

$$|D^\beta f(x)| \le M|\beta|!r^{-|\beta|} \quad \textit{in } D,$$

where

$$M = \frac{\mu}{(1-q)^n}, \qquad\qquad r = (1-q)\min_i |z_i|.$$

Proof. See F. John [10], p. 64. Or an exercise. Hint: Use formula **12.** where x is replaced by $(q, \ldots, q)$.

Remark. From the proposition above it follows

$$c_\alpha = \frac{1}{\alpha!} D^\alpha f(0).$$

Definition. Assume f is defined on a domain $\Omega \subset \mathbb{R}^n$, then f is said to be *real analytic in* $y \in \Omega$ if there are $c_\alpha \in \mathbb{R}$ and if there is a neighbourhood $N(y)$ of y such that

$$f(x) = \sum_\alpha c_\alpha (x-y)^\alpha$$

for all $x \in N(y)$, and the series converges (absolutely) for each $x \in N(y)$. A function f is called *real analytic in* Ω if it is real analytic for each $y \in \Omega$. We will write $f \in C^\omega(\Omega)$ in the case that f is real analytic in the domain Ω. A vector valued function $f(x) = (f_1(x), \ldots, f_m)$ is called real analytic if each coordinate is real analytic.

Proposition A2. (i) *Let* $f \in C^\omega(\Omega)$. *Then* $f \in C^\infty(\Omega)$.

(ii) *Assume* $f \in C^\omega(\Omega)$. *Then for each* $y \in \Omega$ *there exists a neighbourhood* $N(y)$ *and positive constants* M, r *such that*

$$f(x) = \sum_\alpha \frac{1}{\alpha!}(D^\alpha f(y))(x-y)^\alpha$$

for all $x \in N(y)$, *and the series converges (absolutely) for each* $x \in N(y)$, *and*

$$|D^\beta f(x)| \le M|\beta|!r^{-|\beta|}.$$

The proof follows from Proposition A1.

An open set $\Omega \in \mathbb{R}^n$ is called *connected* if Ω is not a union of two nonempty open sets with empty intersection. An open set $\Omega \in \mathbb{R}^n$ is connected if and only if its path connected, see [11], pp. 38, for example. We say that Ω is *path connected* if for any $x,\ y \in \Omega$ there is a continuous curve $\gamma(t) \in \Omega$, $0 \le t \le 1$, with $\gamma(0) = x$ and $\gamma(1) = y$. From the theory of one complex variable we know that a continuation of an analytic function is uniquely determined. The same is true for real analytic functions.

Proposition A3. Assume $f \in C^\omega(\Omega)$ and Ω is connected. Then f is uniquely determined if for one $z \in \Omega$ all $D^\alpha f(z)$ are known.

Proof. See F. John [10], p. 65. Suppose $g,\ h \in C^\omega(\Omega)$ and $D^\alpha g(z) = D^\alpha h(z)$ for every α. Set $f = g - h$ and

$$\begin{aligned}\Omega_1 &= \{x \in \Omega :\ D^\alpha f(x) = 0 \ \text{ for all } \alpha\},\\ \Omega_2 &= \{x \in \Omega :\ D^\alpha f(x) \neq 0 \ \text{ for at least one } \alpha\}.\end{aligned}$$

The set Ω_2 is open since $D^\alpha f$ are continuous in Ω. The set Ω_1 is also open since $f(x) = 0$ in a neighbourhood of $y \in \Omega_1$. This follows from

$$f(x) = \sum_\alpha \frac{1}{\alpha!}(D^\alpha f(y))(x-y)^\alpha.$$

Since $z \in \Omega_1$, i. e., $\Omega_1 \neq \emptyset$, it follows $\Omega_2 = \emptyset$. $\Box$

It was shown in Proposition A2 that derivatives of a real analytic function satisfy estimates. On the other hand it follows, see the next proposition, that a function $f \in C^\infty$ is real analytic if these estimates are satisfied.

Definition. Let $y \in \Omega$ and $M,\ r$ positive constants. Then f is said to be in the class $C_{M,r}(y)$ if $f \in C^\infty$ in a neighbourhood of y and if

$$|D^\beta f(y)| \le M|\beta|!r^{-|\beta|}$$

for all β.

Proposition A4. *$f \in C^\omega(\Omega)$ if and only if $f \in C^\infty(\Omega)$ and for every compact subset $S \subset \Omega$ there are positive constants $M,\ r$ such that*

$$f \in C_{M,r}(y) \quad \textit{for all } y \in S.$$

Proof. See F. John [10], pp. 65-66. We will prove the local version of the proposition, that is, we show it for each fixed $y \in \Omega$. The general version follows from Heine-Borel theorem. Because of Proposition A3 it remains to show that the Taylor series

$$\sum_\alpha \frac{1}{\alpha!} D^\alpha f(y)(x-y)^\alpha$$

converges (absolutely) in a neighbourhood of y and that this series is equal to $f(x)$.

Define a neighbourhood of y by

$$N_d(y) = \{x \in \Omega : \quad |x_1 - y_1| + \ldots + |x_n - y_n| < d\},$$

where d is a sufficiently small positive constant. Set $\Phi(t) = f(y + t(x-y))$. The one-dimensional Taylor theorem says

$$f(x) = \Phi(1) = \sum_{k=0}^{j-1} \frac{1}{k!}\Phi^{(k)}(0) + r_j,$$

where

$$r_j = \frac{1}{(j-1)!}\int_0^1 (1-t)^{j-1}\Phi^{(j)}(t)\ dt.$$

From formula **7.** for directional derivatives it follows for $x \in N_d(y)$ that

$$\frac{1}{j!}\frac{d^j}{dt^j}\Phi(t) = \sum_{|\alpha|=j} \frac{1}{\alpha!} D^\alpha f(y+t(x-y))(x-y)^\alpha .$$

From the assumption and the multinomial formula **3.** we get for $0 \le t \le 1$

$$\begin{aligned}\left|\frac{1}{j!}\frac{d^j}{dt^j}\Phi(t)\right| &\le M \sum_{|\alpha|=j} \frac{|\alpha|!}{\alpha!} r^{-|\alpha|} \left|(x-y)^\alpha\right| \\ &= M r^{-j} \left(|x_1-y_1| + \ldots + |x_n-y_n|\right)^j \\ &\le M \left(\frac{d}{r}\right)^j .\end{aligned}$$

Choose $d > 0$ such that $d < r$, then the Taylor series converges (absolutely) in $N_d(y)$ and it is equal to $f(x)$ since the remainder satisfies, see the above estimate,

$$|r_j| = \left|\frac{1}{(j-1)!}\int_0^1 (1-t)^{j-1}\Phi^j(t)\ dt\right| \le M \left(\frac{d}{r}\right)^j .$$

$\Box$

We recall that the notation $f << F$ (f is majorized by F) was defined in the previous section.

Proposition A5. (i) $f = (f_1, \ldots, f_m) \in C_{M,r}(0)$ *if and only if* $f << (\Phi, \ldots, \Phi)$, *where*

$$\Phi(x) = \frac{Mr}{r - x_1 - \ldots - x_n} .$$

(ii) $f \in C_{M,r}(0)$ *and* $f(0) = 0$ *if and only if*

$$f << (\Phi - M, \ldots, \Phi - M),$$

where

$$\Phi(x) = \frac{M(x_1 + \ldots + x_n)}{r - x_1 - \ldots - x_n} .$$

Proof.

$$D^\alpha \Phi(0) = M|\alpha|! r^{-|\alpha|}.$$

□

Remark. The definition of $f << F$ implies, trivially, that $D^\alpha f << D^\alpha F$.

The next proposition shows that compositions majorize if the involved functions majorize. More precisely, we have

Proposition A6. *Let* $f,\ F:\ \mathbb{R}^n \mapsto \mathbb{R}^m$ *and* $g,\ G$ *maps a neighbourhood of* $0 \in \mathbb{R}^m$ *into* $\mathbb{R}^p$. *Assume all functions* $f(x),\ F(x),\ g(u),\ G(u)$ *are in* C^∞, $f(0) = F(0) = 0$, $f << F$ *and* $g << G$. *Then* $g(f(x)) << G(F(x))$.

Proof. See F. John [10], p. 68. Set

$$h(x) = g(f(x)), \quad H(x) = G(F(x)).$$

For each coordinate h_k of h we have, according to the chain rule,

$$D^\alpha h_k(0) = P_\alpha(\delta^\beta g_l(0), D^\gamma f_j(0)),$$

where P_α are polynomials with *nonnegative* integers as coefficients, P_α are independent on g or f and $\delta := (\partial/\partial u_1, \ldots, \partial/\partial u_m)$. Thus,

$$\begin{aligned} |D^\alpha h_k(0)| &\leq P_\alpha(|\delta^\beta g_l(0)|, |D^\gamma f_j(0)|) \\ &\leq P_\alpha(\delta^\beta G_l(0), D^\gamma F_j(0)) \\ &= D^\alpha H_k(0). \end{aligned}$$

□

Using this result and Proposition A4, which characterizes real analytic functions, it follows that compositions of real analytic functions are real analytic functions again.

Proposition A7. *Assume* $f(x)$ *and* $g(u)$ *are real analytic, then* $g(f(x))$ *is real analytic if* $f(x)$ *is in the domain of definition of* g.

Proof. See F. John [10], p. 68. Assume that f maps a neighbourhood of $y \in \mathbb{R}^n$ in $\mathbb{R}^m$ and g maps a neighbourhood of $v = f(y)$ in $\mathbb{R}^m$. Then $f \in C_{M,r}(y)$ and $g \in C_{\mu,\rho}(v)$ implies

$$h(x) := g(f(x)) \in C_{\mu,\rho r/(mM+\rho)}(y).$$

Once one has shown this inclusion, the proposition follows from Proposition A4. To show the inclusion, we set

$$h(y + x) := g(f(y + x)) \equiv g(v + f(y + x) - f(x)) =: g^*(f^*(x)),$$

where $v = f(y)$ and

$$\begin{aligned} g^*(u) : &= g(v+u) \in C_{\mu,\rho}(0) \\ f^*(x) : &= f(y+x) - f(y) \in C_{M,r}(0). \end{aligned}$$

In the above formulas $v,\ y$ are considered as fixed parameters. From Proposition A5 it follows

$$\begin{aligned} f^*(x) &<< (\Phi - M, \ldots, \Phi - M) =: F \\ g^*(u) &<< (\Psi, \ldots, \Psi) =: G, \end{aligned}$$

where

$$\begin{aligned} \Phi(x) &= \frac{Mr}{r - x_1 - x_2 - \ldots - x_n} \\ \Psi(u) &= \frac{\mu\rho}{\rho - x_1 - x_2 - \ldots - x_n}. \end{aligned}$$

From Proposition A6 we get

$$h(y+x) << (\chi(x), \ldots, \chi(x)) \equiv G(F),$$

where

$$\begin{aligned} \chi(x) &= \frac{\mu\rho}{\rho - m(\Phi(x) - M)} \\ &= \frac{\mu\rho(r - x_1 - \ldots - x_n)}{\rho r - (\rho + mM)(x_1 + \ldots + x_n)} \\ &<< \frac{\mu\rho r}{\rho r - (\rho + mM)(x_1 + \ldots + x_n)} \\ &= \frac{\mu\rho r/(\rho + mM)}{\rho r/(\rho + mM) - (x_1 + \ldots x_n)}. \end{aligned}$$

See an exercise for the "$<<$"-inequality. □

3.6 Exercises

1. Let χ: $\mathbb{R}^n \to \mathbb{R}$ in C^1, $\nabla\chi \neq 0$. Show that for given $x_0 \in \mathbb{R}^n$ there is in a neighbourhood of x_0 a local diffeomorphism $\lambda = \Phi(x)$, $\Phi : (x_1, \ldots, x_n) \mapsto (\lambda_1, \ldots, \lambda_n)$, such that $\lambda_n = \chi(x)$.

2. Show that the differential equation
$$a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} + \text{lower order terms} = 0$$
is elliptic if $ac - b^2 > 0$, parabolic if $ac - b^2 = 0$ and hyperbolic if $ac - b^2 < 0$.

3. Show that in the hyperbolic case there exists a solution of $\phi_x + \mu_1\phi_y = 0$, see equation (3.9), such that $\nabla\phi \neq 0$.

   Hint: Consider an appropriate Cauchy initial value problem.

4. Show equation (3.4).

5. Find the type of
$$Lu := 2u_{xx} + 2u_{xy} + 2u_{yy} = 0$$
and transform this equation into an equation with vanishing mixed derivatives by using the orthogonal mapping (transform to principal axis) $x = Uy$, U orthogonal.

6. Determine the type of the following equation at $(x,y) = (1, 1/2)$.
$$Lu := xu_{xx} + 2yu_{xy} + 2xyu_{yy} = 0.$$

7. Find all C^2-solutions of
$$u_{xx} - 4u_{xy} + u_{yy} = 0.$$
Hint: Transform to principal axis and stretching of axis lead to the wave equation.

8. Oscillations of a beam are described by
$$\begin{aligned} w_x - \frac{1}{E}\sigma_t &= 0 \\ \sigma_x - \rho w_t &= 0, \end{aligned}$$

where σ stresses, w deflection of the beam and $E,\ \rho$ are positive constants.
a) Determine the type of the system.
b) Transform the system into two uncoupled equations, that is, $w,\ \sigma$ occur only in one equation, respectively.
c) Find non-zero solutions.

9. Find nontrivial solutions ($\nabla\chi \neq 0$) of the characteristic equation to

$$x^2u_{xx} - u_{yy} = f(x, y, u, \nabla u),$$

where f is given.

10. Determine the type of

$$u_{xx} - xu_{yx} + u_{yy} + 3u_x = 2x,$$

where $u = u(x, y)$.

11. Transform equation

$$u_{xx} + (1 - y^2)u_{xy} = 0,$$

$u = u(x, y)$, into its normal form.

12. Transform the Tricomi-equation

$$yu_{xx} + u_{yy} = 0,$$

$u = u(x, y)$, where $y < 0$, into its normal form.

13. Transform equation

$$x^2u_{xx} - y^2u_{yy} = 0,$$

$u = u(x, y)$, into its normal form.

14. Show that

$$\lambda = \frac{1}{(1 + |p|^2)^{3/2}}, \quad \Lambda = \frac{1}{(1 + |p|^2)^{1/2}}.$$

are the minimum and maximum of eigenvalues of the matrix (a^{ij}), where

$$a^{ij} = \left(1 + |p|^2\right)^{-1/2}\left(\delta_{ij} - \frac{p_i p_j}{1 + |p|^2}\right).$$

15. Show that Maxwell equations are a hyperbolic system.

16. Consider Maxwell equations and prove that div $E = 0$ and div $H = 0$ for all t if these equations are satisfied for a fixed time t_0.

    Hint. div rot $A = 0$ for each C^2-vector field $A = (A_1, A_2, A_3)$.

17. Assume a characteristic surface $\mathcal{S}(t)$ in $\mathbb{R}^3$ is defined by $\chi(x, y, z, t) = const.$ such that $\chi_t = 0$ and $\chi_z \neq 0$. Show that $\mathcal{S}(t)$ has a nonparametric representation $z = u(x, y, t)$ with $u_t = 0$, that is $\mathcal{S}(t)$ is independent of t.

18. Prove formula (3.22) for the normal on a surface.

19. Prove formula (3.23) for the speed of the surface $\mathcal{S}(t)$.

20. Write the Navier-Stokes system as a system of type (3.24).

21. Show that the following system (linear elasticity, stationary case of (3.25) in the two dimensional case) is elliptic

    $$\mu\triangle u + (\lambda + \mu)\ \mathrm{grad}(\mathrm{div}\ u) + f = 0,$$

    where $u = (u_1, u_2)$. The vector $f = (f_1, f_2)$ is given and λ, μ are positive constants.

22. Discuss the type of the following system in stationary gas dynamics (isentrop flow) in $\mathbb{R}^2$.

    $$\begin{aligned} \rho u u_x + \rho v u_y + a^2 \rho_x &= 0 \\ \rho u v_x + \rho v v_y + a^2 \rho_y &= 0 \\ \rho(u_x + v_y) + u\rho_x + v\rho_y &= 0. \end{aligned}$$

    Here are (u, v) velocity vector, ρ density and $a = \sqrt{p'(\rho)}$ the sound velocity.

23. Show formula **7.** (directional derivative).

    Hint: Induction with respect to m.

24. Let $y = y(x)$ be the solution of:

    $$\begin{aligned} y'(x) &= f(x, y(x)) \\ y(x_0) &= y_0, \end{aligned}$$

where f is real analytic in a neighbourhood of $(x_0, y_0) \in \mathbb{R}^2$. Find the polynomial P of degree 2 such that

$$y(x) = P(x - x_0) + O(|x - x_0|^3)$$

as $x \to x_0$.

25. Let u be the solution of

$$\begin{aligned} \triangle u &= 1 \\ u(x,0) &= u_y(x,0) = 0. \end{aligned}$$

Find the polynomial P of degree 2 such that

$$u(x,y) = P(x,y) + O((x^2 + y^2)^{3/2})$$

as $(x,y) \to (0,0)$.

26. Solve the Cauchy initial value problem

$$\begin{aligned} V_t &= \frac{Mr}{r - s - NV}(1 + N(n-1)V_s) \\ V(s,0) &= 0. \end{aligned}$$

Hint: Multiply the differential equation with $(r - s - NV)$.

27. Write $\triangle^2 u = -u$ as a system of first order.
Hint: $\triangle^2 u \equiv \triangle(\triangle u)$.

28. Write the minimal surface equation

$$\frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{1 + u_x^2 + u_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{u_y}{\sqrt{1 + u_x^2 + u_y^2}}\right) = 0$$

as a system of first order.
Hint: $v_1 := u_x/\sqrt{1 + u_x^2 + u_y^2}$, $v_2 := u_y/\sqrt{1 + u_x^2 + u_y^2}$.

29. Let $f : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ be real analytic in (x_0, y_0). Show that a real analytic solution in a neighbourhood of x_0 of the problem

$$\begin{aligned} y'(x) &= f(x,y) \\ y(x_0) &= y_0 \end{aligned}$$

exists and is equal to the unique $C^1[x_0 - \epsilon, x_0 + \epsilon]$-solution from the Picard-Lindelöf theorem, $\epsilon > 0$ sufficiently small.

30. Show (see the proof of Proposition A7)

$$\frac{\mu\rho(r - x_1 - \ldots - x_n)}{\rho r - (\rho + mM)(x_1 + \ldots + x_n)} << \frac{\mu\rho r}{\rho r - (\rho + mM)(x_1 + \ldots + x_n)}.$$

Hint: Leibniz's rule.

Chapter 4

Hyperbolic equations

Here we consider hyperbolic equations of second order, mainly wave equations.

4.1 One-dimensional wave equation

The one-dimensional wave equation is given by

$$\frac{1}{c^2}u_{tt} - u_{xx} = 0, \tag{4.1}$$

where $u = u(x,t)$ is a scalar function of two variables and c is a positive constant. According to previous considerations, all C^2-solutions of the wave equation are

$$u(x,t) = f(x+ct) + g(x-ct), \tag{4.2}$$

with arbitrary C^2-functions f and g

The *Cauchy initial value problem* for the wave equation is to find a C^2-solution of

$$\begin{aligned}\frac{1}{c^2}u_{tt} - u_{xx} &= 0\\ u(x,0) &= \alpha(x)\\ u_t(x,0) &= \beta(x),\end{aligned}$$

where $\alpha,\ \beta \in C^2(-\infty,\infty)$ are given.

Theorem 4.1. *There exists a unique $C^2(\mathbb{R}\times\mathbb{R})$-solution of the Cauchy initial value problem, and this solution is given by d'Alembert's*[1] *formula*

$$u(x,t)=\frac{\alpha(x+ct)+\alpha(x-ct)}{2}+\frac{1}{2c}\int_{x-ct}^{x+ct}\beta(s)\ ds. \tag{4.3}$$

Proof. Assume there is a solution $u(x,t)$ of the Cauchy initial value problem, then it follows from (4.2) that

$$\begin{aligned} u(x,0) &= f(x)+g(x)=\alpha(x) && (4.4)\\ u_t(x,0) &= cf'(x)-cg'(x)=\beta(x). && (4.5)\end{aligned}$$

From (4.4) we obtain

$$f'(x)+g'(x)=\alpha'(x),$$

which implies, together with (4.5), that

$$\begin{aligned} f'(x) &= \frac{\alpha'(x)+\beta(x)/c}{2}\\ g'(x) &= \frac{\alpha'(x)-\beta(x)/c}{2}.\end{aligned}$$

Then

$$\begin{aligned} f(x) &= \frac{\alpha(x)}{2}+\frac{1}{2c}\int_0^x\beta(s)\ ds+C_1\\ g(x) &= \frac{\alpha(x)}{2}-\frac{1}{2c}\int_0^x\beta(s)\ ds+C_2.\end{aligned}$$

The constants C_1, C_2 satisfy

$$C_1+C_2=f(x)+g(x)-\alpha(x)=0,$$

see (4.4). Thus each C^2-solution of the Cauchy initial value problem is given by d'Alembert's formula. On the other hand, the function $u(x,t)$ defined by the right hand side of (4.3) is a solution of the initial value problem. □

Corollaries. 1. The solution $u(x,t)$ of the initial value problem depends on the values of α at the endpoints of the interval $[x-ct,x+ct]$ and on the values of β on this interval only, see Figure 4.1. The interval $[x-ct,x+ct]$ is called *domain of dependence.*

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[1] d'Alembert, Jean Babtiste le Rond, 1717-1783

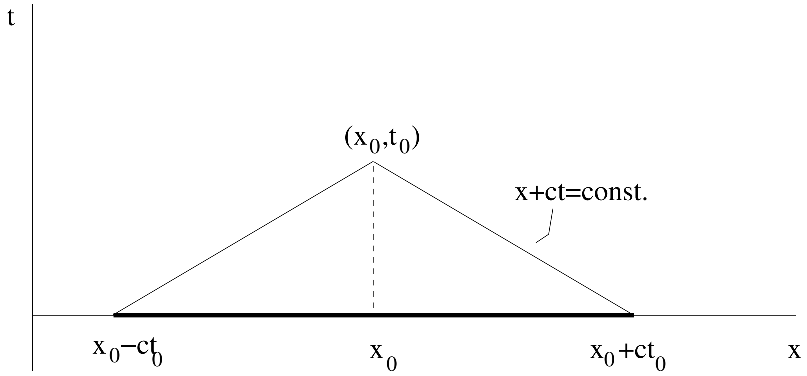

Figure 4.1: Interval of dependence

2. Let P be a point on the x-axis. Then we ask which points (x, t) need values of α or β at P in order to calculate $u(x, t)$? From the d'Alembert formula it follows that this domain is a cone, see Figure 4.2. This set is called *domain of influence*.

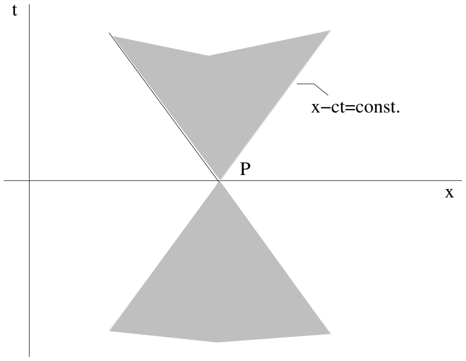

Figure 4.2: Domain of influence

4.2 Higher dimensions

Set

$$\Box u = u_{tt} - c^2 \triangle u, \quad \triangle \equiv \triangle_x = \partial^2/\partial x_1^2 + \ldots + \partial^2/\partial x_n^2,$$

and consider the initial value problem

$$\begin{aligned} \Box u &= 0 \quad \text{in } \mathbb{R}^n \times \mathbb{R} && (4.6)\\ u(x,0) &= f(x) && (4.7)\\ u_t(x,0) &= g(x), && (4.8) \end{aligned}$$

where f and g are given $C^2(\mathbb{R}^2)$-functions.

By using spherical means and the above d'Alembert formula we will derive a formula for the solution of this initial value problem.

Method of spherical means

Define the spherical mean for a C^2-solution $u(x,t)$ of the initial value problem by

$$M(r,t) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} u(y,t)\ dS_y, \tag{4.9}$$

where

$$\omega_n = (2\pi)^{n/2}/\Gamma(n/2)$$

is the area of the n-dimensional sphere, $\omega_n r^{n-1}$ is the area of a sphere with radius r.

From the mean value theorem of the integral calculus we obtain the function $u(x,t)$ for which we are looking at by

$$u(x,t) = \lim_{r\to 0} M(r,t). \tag{4.10}$$

Using the initial data, we have

$$\begin{aligned} M(r,0) &= \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} f(y)\ dS_y =: F(r) && (4.11)\\ M_t(r,0) &= \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} g(y)\ dS_y =: G(r), && (4.12) \end{aligned}$$

which are the spherical means of f and g.

The next step is to derive a partial differential equation for the spherical mean. From definition (4.9) of the spherical mean we obtain, after the mapping $\xi = (y-x)/r$, where x and r are fixed,

$$M(r,t) = \frac{1}{\omega_n} \int_{\partial B_1(0)} u(x + r\xi, t)\ dS_\xi.$$

It follows

$$\begin{aligned} M_r(r,t) &= \frac{1}{\omega_n}\int_{\partial B_1(0)} \sum_{i=1}^n u_{y_i}(x+r\xi,t)\xi_i\ dS_\xi \\ &= \frac{1}{\omega_n r^{n-1}}\int_{\partial B_r(x)} \sum_{i=1}^n u_{y_i}(y,t)\xi_i\ dS_y. \end{aligned}$$

Integration by parts yields

$$\frac{1}{\omega_n r^{n-1}}\int_{B_r(x)} \sum_{i=1}^n u_{y_i y_i}(y,t)\ dy$$

since $\xi \equiv (y-x)/r$ is the exterior normal at $\partial B_r(x)$. Assume u is a solution of the wave equation, then

$$\begin{aligned} r^{n-1}M_r &= \frac{1}{c^2\omega_n}\int_{B_r(x)} u_{tt}(y,t)\ dy \\ &= \frac{1}{c^2\omega_n}\int_0^r \int_{\partial B_c(x)} u_{tt}(y,t)\ dS_y dc. \end{aligned}$$

The previous equation follows by using spherical coordinates. Consequently

$$\begin{aligned} (r^{n-1}M_r)_r &= \frac{1}{c^2\omega_n}\int_{\partial B_r(x)} u_{tt}(y,t)\ dS_y \\ &= \frac{r^{n-1}}{c^2}\frac{\partial^2}{\partial t^2}\left(\frac{1}{\omega_n r^{n-1}}\int_{\partial B_r(x)} u(y,t)\ dS_y\right) \\ &= \frac{r^{n-1}}{c^2}M_{tt}. \end{aligned}$$

Thus we arrive at the differential equation

$$(r^{n-1}M_r)_r = c^{-2}r^{n-1}M_{tt},$$

which can be written as

$$M_{rr} + \frac{n-1}{r}M_r = c^{-2}M_{tt}. \tag{4.13}$$

This equation (4.13) is called *Euler-Poisson-Darboux equation.*

4.2.1 Case n=3

The Euler-Poisson-Darboux equation in this case is

$$(rM)_{rr}=c^{-2}(rM)_{tt}.$$

Thus rM is the solution of the one-dimensional wave equation with initial data

$$(rM)(r,0)=rF(r)\quad (rM)_t(r,0)=rG(r). \tag{4.14}$$

From the d'Alembert formula we get formally

$$\begin{aligned} M(r,t) &= \frac{(r+ct)F(r+ct)+(r-ct)F(r-ct)}{2r}\\ &\quad +\frac{1}{2cr}\int_{r-ct}^{r+ct}\xi G(\xi)\ d\xi. \end{aligned} \tag{4.15}$$

The right hand side of the previous formula is well defined if the domain of dependence $[x-ct,x+ct]$ is a subset of $(0,\infty)$. We can extend F and G to F_0 and G_0 which are defined on $(-\infty,\infty)$ such that rF_0 and rG_0 are $C^2(\mathbb{R})$-functions as follows. Set

$$F_0(r)=\left\{\begin{array}{rcl} F(r) & : & r>0\\ f(x) & : & r=0\\ F(-r) & : & r<0 \end{array}\right.$$

The function $G_0(r)$ is given by the same definition where F and f are replaced by G and g, respectively.

Lemma. $rF_0(r),\ rG_0(r)\in C^2(\mathbb{R}^2)$.

Proof. From definition of $F(r)$ and $G(r)$, $r>0$, it follows from the mean value theorem

$$\lim_{r\to+0}F(r)=f(x),\quad \lim_{r\to+0}G(r)=g(x).$$

Thus $rF_0(r)$ and $rG_0(r)$ are $C(\mathbb{R})$-functions. These functions are also in

$C^1(\mathbb{R})$. This follows since F_0 and G_0 are in $C^1(\mathbb{R})$. We have, for example,

$$\begin{aligned}
F'(r) &= \frac{1}{\omega_n}\int_{\partial B_1(0)} \sum_{j=1}^n f_{y_j}(x+r\xi)\xi_j\ dS_\xi \\
F'(+0) &= \frac{1}{\omega_n}\int_{\partial B_1(0)} \sum_{j=1}^n f_{y_j}(x)\xi_j\ dS_\xi \\
&= \frac{1}{\omega_n}\sum_{j=1}^n f_{y_j}(x)\int_{\partial B_1(0)} n_j\ dS_\xi \\
&= 0.
\end{aligned}$$

Then, $rF_0(r)$ and $rG_0(r)$ are in $C^2(\mathbb{R})$, provided F'' and G'' are bounded as $r \to +0$. This property follows from

$$F''(r) = \frac{1}{\omega_n}\int_{\partial B_1(0)} \sum_{i,j=1}^n f_{y_iy_j}(x+r\xi)\xi_i\xi_j\ dS_\xi.$$

Thus

$$F''(+0) = \frac{1}{\omega_n}\sum_{i,j=1}^n f_{y_iy_j}(x)\int_{\partial B_1(0)} n_in_j\ dS_\xi.$$

We recall that $f, g \in C^2(\mathbb{R}^2)$ by assumption. □

The solution of the above initial value problem, where F and G are replaced by F_0 and G_0, respectively, is

$$\begin{aligned}
M_0(r,t) &= \frac{(r+ct)F_0(r+ct)+(r-ct)F_0(r-ct)}{2r} \\
&\quad +\frac{1}{2cr}\int_{r-ct}^{r+ct} \xi G_0(\xi)\ d\xi.
\end{aligned}$$

Since F_0 and G_0 are even functions, we have

$$\int_{r-ct}^{ct-r} \xi G_0(\xi)\ d\xi = 0.$$

Thus

$$\begin{aligned}
M_0(r,t) &= \frac{(r+ct)F_0(r+ct)-(ct-r)F_0(ct-r)}{2r} \\
&\quad +\frac{1}{2cr}\int_{ct-r}^{ct+r} \xi G_0(\xi)\ d\xi, \qquad (4.16)
\end{aligned}$$

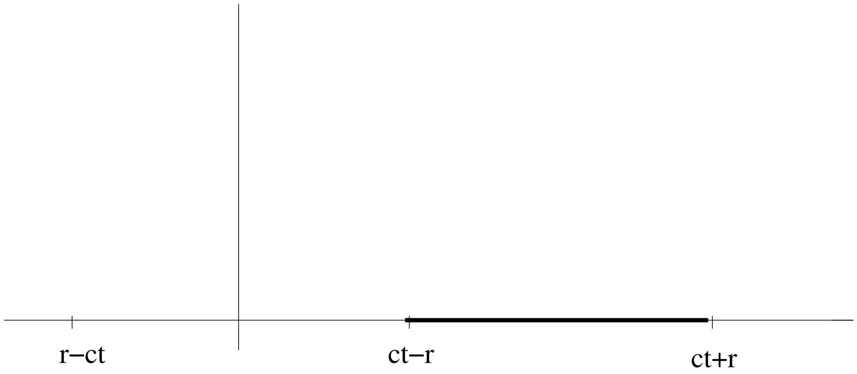

Figure 4.3: Changed domain of integration

see Figure 4.3. For fixed $t > 0$ and $0 < r < ct$ it follows that $M_0(r,t)$ is the solution of the initial value problem with given initially data (4.14) since $F_0(s) = F(s)$, $G_0(s) = G(s)$ if $s > 0$. Since for fixed $t > 0$

$$u(x,t) = \lim_{r\to 0} M_0(r,t),$$

it follows from d'Hospital's rule that

$$\begin{aligned} u(x,t) &= ctF'(ct) + F(ct) + tG(ct) \\ &= \frac{d}{dt}\left(tF(ct)\right) + tG(ct). \end{aligned}$$

Theorem 4.2. *Assume $f \in C^3(\mathbb{R}^3)$ and $g \in C^2(\mathbb{R}^3)$ are given. Then there exists a unique solution $u \in C^2(\mathbb{R}^3 \times [0,\infty))$ of the initial value problem (4.6)-(4.7), where $n = 3$, and the solution is given by the Poisson's formula*

$$\begin{aligned} u(x,t) &= \frac{1}{4\pi c^2}\frac{\partial}{\partial t}\left(\frac{1}{t}\int_{\partial B_{ct}(x)} f(y)\ dS_y\right) \\ &\quad + \frac{1}{4\pi c^2 t}\int_{\partial B_{ct}(x)} g(y)\ dS_y. \end{aligned} \tag{4.17}$$

Proof. Above we have shown that a C^2-solution is given by Poisson's formula. Under the additional assumption $f \in C^3$ it follows from Poisson's formula that this formula defines a solution which is in C^2, see F. John [10], p. 129. $\square$

Corollary. From Poisson's formula we see that the domain of dependence for $u(x,t_0)$ is the intersection of the cone defined by $|y-x|=c|t-t_0|$ with the hyperplane defined by $t=0$, see Figure 4.4

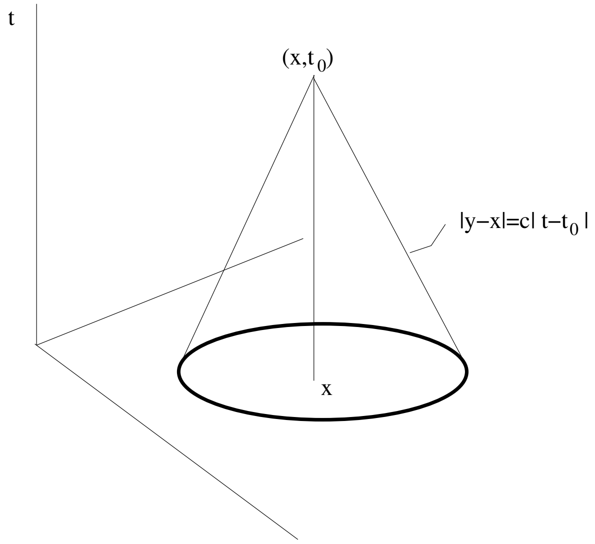

Figure 4.4: Domain of dependence, case $n=3$

4.2.2 Case $n=2$

Consider the initial value problem

$$\begin{aligned} v_{xx}+v_{yy} &= c^{-2}v_{tt} && (4.18)\\ v(x,y,0) &= f(x,y) && (4.19)\\ v_t(x,y,0) &= g(x,y), && (4.20) \end{aligned}$$

where $f\in C^3,\ g\in C^2$.

Using the formula for the solution of the three-dimensional initial value problem we will derive a formula for the two-dimensional case. The following consideration is called *Hadamard's method of decent*.

Let $v(x,y,t)$ be a solution of (4.18)-(4.20), then

$$u(x,y,z,t):=v(x,y,t)$$

is a solution of the three-dimensional initial value problem with initial data $f(x,y)$, $g(x,y)$, independent of z, since u satisfies (4.18)-(4.20). Hence, since $u(x,y,z,t) = u(x,y,0,t) + u_z(x,y,\delta z,t)z$, $0 < \delta < 1$, and $u_z = 0$, we have

$$v(x,y,t) = u(x,y,0,t).$$

Poisson's formula in the three-dimensional case implies

$$\begin{aligned} v(x,y,t) &= \frac{1}{4\pi c^2}\frac{\partial}{\partial t}\left(\frac{1}{t}\int_{\partial B_{ct}(x,y,0)} f(\xi,\eta)\ dS\right) \\ &\quad + \frac{1}{4\pi c^2 t}\int_{\partial B_{ct}(x,y,0)} g(\xi,\eta)\ dS. \end{aligned} \tag{4.21}$$

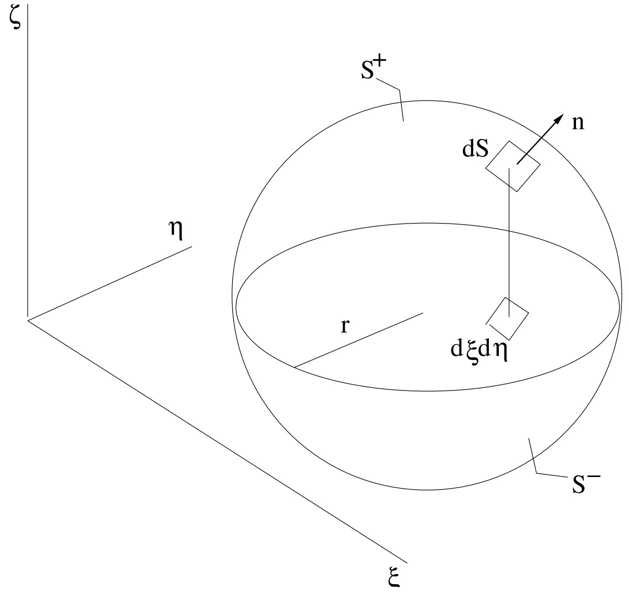

Figure 4.5: Domains of integration

The integrands are independent on ζ. The surface S is defined by $\chi(\xi,\eta,\zeta) := (\xi-x)^2 + (\eta-y)^2 + \zeta^2 - c^2t^2 = 0$. Then the exterior normal n at S is $n = \nabla\chi/|\nabla\chi|$ and the surface element is given by $dS = (1/|n_3|)d\xi d\eta$, where the third coordinate of n is

$$n_3 = \pm\frac{\sqrt{c^2t^2 - (\xi-x)^2 - (\eta-y)^2}}{ct}.$$

The positive sign applies on S^+, where $\zeta > 0$ and the sign is negative on S^- where $\zeta < 0$, see Figure 4.5. We have $S = S^+ \cup \overline{S^-}$.

Set $\rho = \sqrt{(\xi - x)^2 + (\eta - y)^2}$. Then it follows from (4.21)

Theorem 4.3. *The solution of the Cauchy initial value problem (4.18)-(4.20) is given by*

$$\begin{aligned} v(x,y,t) &= \frac{1}{2\pi c}\frac{\partial}{\partial t}\int_{B_{ct}(x,y)} \frac{f(\xi,\eta)}{\sqrt{c^2t^2-\rho^2}}\ d\xi d\eta \\ &\quad + \frac{1}{2\pi c}\int_{B_{ct}(x,y)} \frac{g(\xi,\eta)}{\sqrt{c^2t^2-\rho^2}}\ d\xi d\eta. \end{aligned}$$

Corollary. In contrast to the three dimensional case, the domain of dependence is here the disk $B_{ct_o}(x_0, y_0)$ and not the boundary only. Therefore, see formula of Theorem 4.3, if $f,\ g$ have supports in a compact domain $D \subset \mathbb{R}^2$, then these functions have influence on the value $v(x,y,t)$ for *all* time $t > T$, T sufficiently large.

4.3 Inhomogeneous equation

Here we consider the initial value problem

$$\begin{aligned} \Box u &= w(x,t) \ \text{ on } x \in \mathbb{R}^n,\ t \in \mathbb{R} && (4.22) \\ u(x,0) &= f(x) && (4.23) \\ u_t(x,0) &= g(x), && (4.24) \end{aligned}$$

where $\Box u := u_{tt} - c^2 \triangle u$. We assume $f \in C^3$, $g \in C^2$ and $w \in C^1$, which are given.

Set $u = u_1 + u_2$, where u_1 is a solution of problem (4.22)-(4.24) with $w := 0$ and u_2 is the solution where $f = 0$ and $g = 0$ in (4.22)-(4.24). Since we have explicit solutions u_1 in the cases $n = 1$, $n = 2$ and $n = 3$, it remains to solve

$$\begin{aligned} \Box u &= w(x,t) \ \text{ on } x \in \mathbb{R}^n,\ t \in \mathbb{R} && (4.25) \\ u(x,0) &= 0 && (4.26) \\ u_t(x,0) &= 0. && (4.27) \end{aligned}$$

The following method is called *Duhamel's principle* which can be considered as a generalization of the method of variations of constants in the theory of ordinary differential equations.

To solve this problem, we make the ansatz

$$u(x,t)=\int_0^t\ v(x,t,s)\ ds, \tag{4.28}$$

where v is a function satisfying

$$\Box v=0\ \text{ for all } s \tag{4.29}$$

and

$$v(x,s,s)=0. \tag{4.30}$$

From ansatz (4.28) and assumption (4.30) we get

$$\begin{aligned} u_t &= v(x,t,t)+\int_0^t\ v_t(x,t,s)\ ds,\\ &= \int_0^t\ v_t(x,t,s). \end{aligned} \tag{4.31}$$

It follows $u_t(x,0)=0$. The initial condition $u(x,t)=0$ is satisfied because of the ansatz (4.28). From (4.31) and ansatz (4.28) we see that

$$\begin{aligned} u_{tt} &= v_t(x,t,t)+\int_0^t\ v_{tt}(x,t,s)\ ds,\\ \triangle_x u &= \int_0^t\ \triangle_x v(x,t,s)\ ds. \end{aligned}$$

Therefore, since u is an ansatz for (4.25)-(4.27),

$$\begin{aligned} u_{tt}-c^2\triangle_x u &= v_t(x,t,t)+\int_0^t (\Box v)(x,t,s)\ ds\\ &= w(x,t). \end{aligned}$$

Thus necessarily $v_t(x,t,t)=w(x,t)$, see (4.29). We have seen that the ansatz provides a solution of (4.25)-(4.27) if for all s

$$\Box v=0,\quad v(x,s,s)=0,\quad v_t(x,s,s)=w(x,s). \tag{4.32}$$

Let $v^*(x,t,s)$ be a solution of

$$\Box v=0,\quad v(x,0,s)=0,\quad v_t(x,0,s)=w(x,s), \tag{4.33}$$

then

$$v(x,t,s):=v^*(x,t-s,s)$$

is a solution of (4.32). In the case $n = 3$, where v^* is given by, see Theorem 4.2,

$$v^*(x,t,s) = \frac{1}{4\pi c^2 t} \int_{\partial B_{ct}(x)} w(\xi, s)\ dS_\xi.$$

Then

$$\begin{aligned} v(x,t,s) &= v^*(x, t-s, s) \\ &= \frac{1}{4\pi c^2 (t-s)} \int_{\partial B_{c(t-s)}(x)} w(\xi, s)\ dS_\xi. \end{aligned}$$

from ansatz (4.28) it follows

$$\begin{aligned} u(x,t) &= \int_0^t v(x,t,s)\ ds \\ &= \frac{1}{4\pi c^2} \int_0^t \int_{\partial B_{c(t-s)}(x)} \frac{w(\xi, s)}{t-s}\ dS_\xi ds. \end{aligned}$$

Changing variables by $\tau = c(t-s)$ yields

$$\begin{aligned} u(x,t) &= \frac{1}{4\pi c^2} \int_0^{ct} \int_{\partial B_\tau(x)} \frac{w(\xi, t-\tau/c)}{\tau}\ dS_\xi d\tau \\ &= \frac{1}{4\pi c^2} \int_{B_{ct}(x)} \frac{w(\xi, t-r/c)}{r}\ d\xi, \end{aligned}$$

where $r = |x - \xi|$.

Formulas for the cases $n = 1$ and $n = 2$ follow from formulas for the associated homogeneous equation with inhomogeneous initial values for these cases.

Theorem 4.4. *The solution of*

$$\Box u = w(x,t), \quad u(x,0) = 0, \quad u_t(x,0) = 0,$$

where $w \in C^1$, is given by:

Case $n = 3$:

$$u(x,t) = \frac{1}{4\pi c^2} \int_{B_{ct}(x)} \frac{w(\xi, t-r/c)}{r}\ d\xi,$$

where $r = |x - \xi|$, $x = (x_1, x_2, x_3)$, $\xi = (\xi_1, \xi_2, \xi_3)$.

Case $n = 2$:

$$u(x,t) = \frac{1}{4\pi c}\int_0^t \left(\int_{B_{c(t-\tau)}(x)} \frac{w(\xi,\tau)}{\sqrt{c^2(t-\tau)^2 - r^2}}\ d\xi \right)\ d\tau,$$

$x = (x_1, x_2)$, $\xi = (\xi_1, \xi_2)$.

Case $n = 1$:

$$u(x,t) = \frac{1}{2c}\int_0^t \left(\int_{x-c(t-\tau)}^{x+c(t-\tau)} w(\xi,\tau)\ d\xi \right)\ d\tau.$$

Remark. The integrand on the right in formula for $n = 3$ is called *retarded potential.* The integrand is taken not at t, it is taken at an *earlier* time $t - r/c$.

4.4 A method of Riemann

Riemann's method provides a formula for the solution of the following Cauchy initial value problem for a hyperbolic equation of second order in two variables. Let

$$\mathcal{S}: \quad x = x(t), y = y(t), \quad t_1 \le t \le t_2,$$

be a regular curve in $\mathbb{R}^2$, that is, we assume $x,\ y \in C^1[t_1, t_2]$ and $x'^2 + y'^2 \neq 0$. Set

$$Lu := u_{xy} + a(x,y)u_x + b(x,y)u_y + c(x,y)u,$$

where $a,\ b \in C^1$ and $c,\ f \in C$ in a neighbourhood of $\mathcal{S}$. Consider the initial value problem

$$\begin{aligned} Lu &= f(x,y) && (4.34)\\ u_0(t) &= u(x(t), y(t)) && (4.35)\\ p_0(t) &= u_x(x(t), y(t)) && (4.36)\\ q_0(t) &= u_y(x(t), y(t)), && (4.37) \end{aligned}$$

where $f \in C$ in a neighbourhood of $\mathcal{S}$ and $u_0,\ p_0,\ q_0 \in C^1$ are given.

We assume:

(i) $u_0'(t) = p_0(t)x'(t) + q_0(t)y'(t)$ (strip condition),

(ii) $\mathcal{S}$ is not a characteristic curve. Moreover assume that the characteristic curves, which are lines here and are defined by $x = const.$ and $y = const.$, have at most one point of intersection with $\mathcal{S}$, and such a point is not a touching point, i. e., tangents of the characteristic and $\mathcal{S}$ are different at this point.

We recall that the characteristic equation to (4.34) is $\chi_x\chi_y = 0$ which is satisfied if $\chi_x(x,y) = 0$ or $\chi_y(x,y) = 0$. One family of characteristics associated to these first partial differential of first order is defined by $x'(t) = 1,\ y'(t) = 0$, see Chapter 2.

Assume $u,\ v \in C^1$ and that $u_{xy},\ v_{xy}$ exist and are continuous. Define the adjoint differential expression by

$$Mv = v_{xy} - (av)_x - (bv)_y + cv.$$

We have

$$2(vLu - uMv) = (u_xv - v_xu + 2buv)_y + (u_yv - v_yu + 2auv)_x. \tag{4.38}$$

Set

$$\begin{aligned} P &= -(u_xv - x_xu + 2buv) \\ Q &= u_yv - v_yu + 2auv. \end{aligned}$$

From (4.38) it follows for a domain $\Omega \in \mathbb{R}^2$

$$\begin{aligned} 2\int_\Omega (vLu - uMv)\ dxdy &= \int_\Omega (-P_y + Q_x)\ dxdy \\ &= \oint Pdx + Qdy, \end{aligned} \tag{4.39}$$

where integration in the line integral is anticlockwise. The previous equation follows from Gauss theorem or after integration by parts:

$$\int_\Omega (-P_y + Q_x)\ dxdy = \int_{\partial\Omega} (-Pn_2 + Qn_1)\ ds,$$

where $n = (dy/ds, -dx/ds)$, s arc length, $(x(s), y(s))$ represents $\partial\Omega$.

Assume u is a solution of the initial value problem (4.34)-(4.37) and suppose that v satisfies

$$Mv = 0 \ \text{ in } \Omega.$$

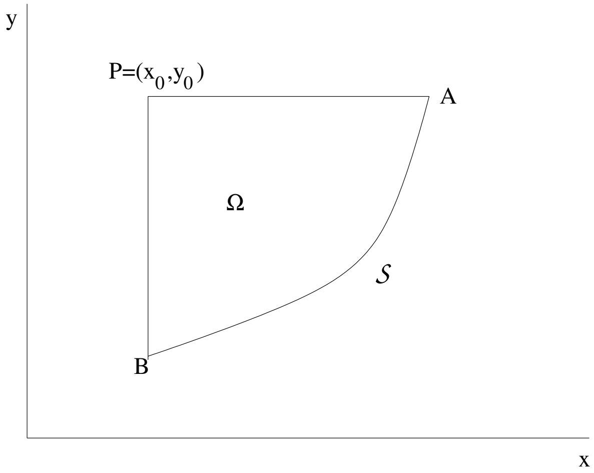

Figure 4.6: Riemann's method, domain of integration

Then, if we integrate over a domain Ω as shown in Figure 4.6, it follows from (4.39) that

$$2\int_\Omega vf\ dxdy=\int_{BA} Pdx+Qdy+\int_{AP} Pdx+Qdy+\int_{PB} Pdx+Qdy. \quad (4.40)$$

The line integral from B to A is known from initial data, see the definition of P and Q.

Since

$$u_xv-v_xu+2buv=(uv)_x+2u(bv-v_x),$$

it follows

$$\begin{aligned}\int_{AP} Pdx+Qdy &= -\int_{AP}((uv)_x+2u(bv-v_x))\ dx\\ &= -(uv)(P)+(uv)(A)-\int_{AP} 2u(bv-v_x)\ dx.\end{aligned}$$

By the same reasoning we obtain for the third line integral

$$\begin{aligned}\int_{PB} Pdx+Qdy &= \int_{PB}((uv)_y+2u(av-v_y))\ dy\\ &= (uv)(B)-(uv)(P)+\int_{PB} 2u(av-v_y)\ dy.\end{aligned}$$

Combining these equations with (4.39), we get

$$\begin{aligned} 2v(P)u(P) &= \int_{BA} (u_x v - v_x + 2buv)\ dx - (u_y v - v_y u + 2auv)\ dy \\ &\quad +u(A)v(A) + u(B)v(B) + 2\int_{AP} u(bv - v_x)\ dx \\ &\quad +2\int_{PB} u(av - v_y)\ dy - 2\int_\Omega fv\ dxdy. \end{aligned} \tag{4.41}$$

Let v be a solution of the initial value problem, see Figure 4.7 for the definition of domain $D(P)$,

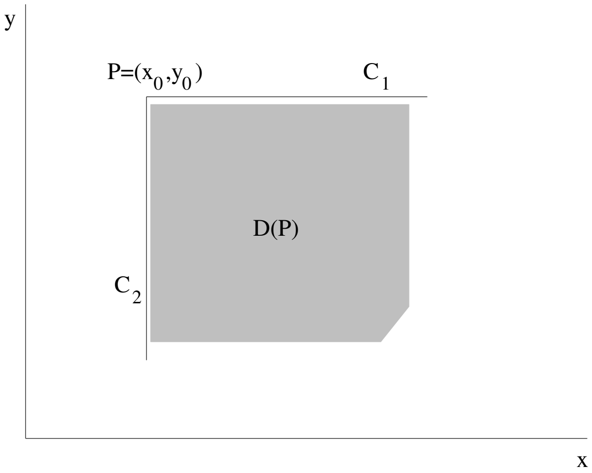

Figure 4.7: Definition of Riemann's function

$$\begin{aligned} Mv &= 0 \ \text{ in } D(P) && (4.42) \\ bv - v_x &= 0 \ \text{ on } C_1 && (4.43) \\ av - v_y &= 0 \ \text{ on } C_2 && (4.44) \\ v(P) &= 1. && (4.45) \end{aligned}$$

Assume v satisfies (4.42)-(4.45), then

$$\begin{aligned} 2u(P) &= u(A)v(A) + u(B)v(B) - 2\int_\Omega fv\ dxdy \\ &= \int_{BA} (u_x v - v_x + 2buv)\ dx - (u_y v - v_y u + 2auv)\ dy, \end{aligned}$$

where the right hand side is known from given data.

A function $v = v(x, y; x_0, y_0)$ satisfying (4.42)-(4.45) is called *Riemann's function.*

Remark. Set $w(x, y) = v(x, y; x_0, y_0)$ for fixed $x_0,\ y_0$. Then (4.42)-(4.45) imply

$$\begin{aligned} w(x, y_0) &= \exp\left(\int_{x_0}^{x} b(\tau, y_0)\ d\tau\right) \quad \text{on } C_1, \\ w(x_0, y) &= \exp\left(\int_{y_0}^{y} a(x_0, \tau)\ d\tau\right) \quad \text{on } C_2. \end{aligned}$$

Examples

1. $u_{xy} = f(x, y)$, then a Riemann function is $v(x, y) \equiv 1$.

2. Consider the telegraph equation of Chapter 3

$$\varepsilon\mu u_{tt} = c^2 \triangle_x u - \lambda\mu u_t,$$

where u stands for one coordinate of electric or magnetic field. Introducing

$$u = w(x, t)\mathrm{e}^{\kappa t},$$

where $\kappa = -\lambda/(2\varepsilon)$, we arrive at

$$w_{tt} = \frac{c^2}{\varepsilon\mu}\triangle_x w - \frac{\lambda^2}{4\epsilon^2}.$$

Stretching the axis and transform the equation to the normal form we get finally the following equation, the new function is denoted by u and the new variables are denoted by $x,\ y$ again,

$$u_{xy} + cu = 0,$$

with a positive constant c. We make the ansatz for a Riemann function

$$v(x, y; x_0, y_0) = w(s), \quad s = (x - x_0)(y - y_0)$$

and obtain

$$sw'' + w' + cw = 0.$$

Substitution $\sigma = \sqrt{4cs}$ leads to Bessel's differential equation

$$\sigma^2 z''(\sigma) + \sigma z'(\sigma) + \sigma^2 z(\sigma) = 0,$$

where $z(\sigma) = w(\sigma^2/(4c))$. A solution is

$$J_0(\sigma) = J_0\left(\sqrt{4c(x-x_0)(y-y_0)}\right)$$

which defines a Riemann function since $J_0(0) = 1$.

Remark. Bessel's differential equation is

$$x^2 y''(x) + xy'(x) + (x^2 - n^2)y(x) = 0,$$

where $n \in \mathbb{R}$. If $n \in \mathbb{N} \cup \{0\}$, then solutions are given by Bessel functions. One of the two linearly independent solutions is bounded at 0. This bounded solution is the Bessel function $J_n(x)$ of first kind and of order n, see [1], for example.

4.5 Initial-boundary value problems

In previous sections we looked at solutions defined for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. In this and in the following section we seek solutions $u(x,t)$ defined in a bounded domain $\Omega \subset \mathbb{R}^n$ and for all $t \in \mathbb{R}$ and which satisfy additional boundary conditions on $\partial\Omega$.

4.5.1 Oscillation of a string

Let $u(x,t)$, $x \in [a,b]$, $t \in \mathbb{R}$, be the deflection of a string, see Figure 1.4 from Chapter 1. Assume the deflection occurs in the (x,u)-plane. This problem is governed by the initial-boundary value problem

$$\begin{aligned} u_{tt}(x,t) &= u_{xx}(x,t) \text{ on } (0,l) && (4.46)\\ u(x,0) &= f(x) && (4.47)\\ u_t(x,0) &= g(x) && (4.48)\\ u(0,t) &= u(l,t) = 0. && (4.49) \end{aligned}$$

Assume the initial data f, g are sufficiently regular. This implies compatibility conditions $f(0) = f(l) = 0$ and $g(0) = g(l)$.

Fourier's method

To find solutions of differential equation (4.46) we make the *separation of variables* ansatz

$$u(x,t) = v(x)w(t).$$

Inserting the ansatz into (4.46) we obtain

$$v(x)w''(t) = v''(x)w(t),$$

or, if $v(x)w(t) \neq 0$,

$$\frac{w''(t)}{w(t)} = \frac{v''(x)}{v(x)}.$$

It follows, provided $v(x)w(t)$ is a solution of differential equation (4.46) and $v(x)w(t) \neq 0$,

$$\frac{w''(t)}{w(t)} = const. =: -\lambda$$

and

$$\frac{v''(x)}{v(x)} = -\lambda$$

since $x,\ t$ are independent variables.

Assume $v(0) = v(l) = 0$, then $v(x)w(t)$ satisfies the boundary condition (4.49). Thus we look for solutions of the eigenvalue problem

$$\begin{aligned} -v''(x) &= \lambda v(x) \ \text{ in } (0,l) && (4.50)\\ v(0) &= v(l) = 0, && (4.51) \end{aligned}$$

which has the eigenvalues

$$\lambda_n = \left(\frac{\pi}{l}n\right)^2, \quad n = 1, 2, \ldots,$$

and associated eigenfunctions are

$$v_n = \sin\left(\frac{\pi}{l}nx\right).$$

Solutions of

$$-w''(t) = \lambda_n w(t)$$

are

$$\sin(\sqrt{\lambda_n}t), \quad \cos(\sqrt{\lambda_n}t).$$

Set

$$w_n(t) = \alpha_n \cos(\sqrt{\lambda_n}t) + \beta_n \sin(\sqrt{\lambda_n}t),$$

where $\alpha_n,\ \beta_n \in \mathbb{R}$. It is easily seen that $w_n(t)v_n(x)$ is a solution of differential equation (4.46), and, since (4.46) is linear and homogeneous, also (principle of superposition)

$$u_N = \sum_{n=1}^{N} w_n(t)v_n(x)$$

which satisfies the differential equation (4.46) and the boundary conditions (4.49). Consider the formal solution of (4.46), (4.49)

$$u(x,t) = \sum_{n=1}^{\infty} \left(\alpha_n \cos(\sqrt{\lambda_n}t) + \beta_n \sin(\sqrt{\lambda_n}t)\right) \sin\left(\sqrt{\lambda_n}x\right). \qquad (4.52)$$

"Formal" means that we know here neither that the right hand side converges nor that it is a solution of the initial-boundary value problem. Formally, the unknown coefficients can be calculated from initial conditions (4.47), (4.48) as follows. We have

$$u(x,0) = \sum_{n=1}^{\infty} \alpha_n \sin(\sqrt{\lambda_n}x) = f(x).$$

Multiplying this equation by $\sin(\sqrt{\lambda_k}x)$ and integrate over $(0,l)$, we get

$$\alpha_n \int_0^l \sin^2(\sqrt{\lambda_k}x)\ dx = \int_0^l f(x)\sin(\sqrt{\lambda_k}x)\ dx.$$

We recall that

$$\int_0^l \sin(\sqrt{\lambda_n}x)\sin(\sqrt{\lambda_k}x)\ dx = \frac{l}{2}\delta_{nk}.$$

Then

$$\alpha_k = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{\pi k}{l}x\right)\ dx. \qquad (4.53)$$

By the same argument it follows from

$$u_t(x,0) = \sum_{n=1}^{\infty} \beta_n\sqrt{\lambda_n}\sin(\sqrt{\lambda_n}x) = g(x)$$

that

$$\beta_k = \frac{2}{k\pi}\int_0^l g(x)\sin\left(\frac{\pi k}{l}x\right)\ dx. \qquad (4.54)$$

Under additional assumptions $f \in C_0^4(0,l)$, $g \in C_0^3(0,l)$ it follows that the right hand side of (4.52), where α_n, β_n are given by (4.53) and (4.54), respectively, defines a classical solution of (4.46)-(4.49) since under these assumptions the series for u and the formal differentiate series for u_t, u_{tt}, u_x, u_{xx} converges uniformly on $0 \leq x \leq l$, $0 \leq t \leq T$, $0 < T < \infty$ fixed, see an exercise.

4.5.2 Oscillation of a membrane

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. We consider the initial-boundary value problem

$$\begin{aligned} u_{tt}(x,t) &= \triangle_x u \quad \text{in } \Omega \times \mathbb{R}, && (4.55) \\ u(x,0) &= f(x), \quad x \in \overline{\Omega}, && (4.56) \\ u_t(x,0) &= g(x), \quad x \in \overline{\Omega}, && (4.57) \\ u(x,t) &= 0 \quad \text{on } \partial\Omega \times \mathbb{R}. && (4.58) \end{aligned}$$

As in the previous subsection for the string, we make the ansatz (separation of variables)

$$u(x,t) = w(t)v(x)$$

which leads to the eigenvalue problem

$$\begin{aligned} -\triangle v &= \lambda v \quad \text{in } \Omega, && (4.59) \\ v &= 0 \quad \text{on } \partial\Omega. && (4.60) \end{aligned}$$

Let λ_n are the eigenvalues of (4.59), (4.60) and v_n a complete associated orthonormal system of eigenfunctions. We assume Ω is sufficiently regular such that the eigenvalues are countable, which is satisfied in the following examples. Then the formal solution of the above initial-boundary value problem is

$$u(x,t) = \sum_{n=1}^{\infty} \left(\alpha_n \cos(\sqrt{\lambda_n} t) + \beta_n \sin(\sqrt{\lambda_n} t) \right) v_n(x),$$

where

$$\begin{aligned} \alpha_n &= \int_\Omega f(x) v_n(x) \, dx \\ \beta_n &= \frac{1}{\sqrt{\lambda_n}} \int_\Omega g(x) v_n(x) \, dx. \end{aligned}$$

Remark. In general, eigenvalues of (4.59), (4.59) are not known explicitly. There are numerical methods to calculate these values. In some special cases, see next examples, these values are known.

Examples

1. Rectangle membrane. Let

$$\Omega = (0,a) \times (0,b).$$

Using the method of separation of variables, we find all eigenvalues of (4.59), (4.60) which are given by

$$\lambda_{kl} = \sqrt{\frac{k^2}{a^2} + \frac{l^2}{b^2}}, \quad k,l = 1,2,\ldots$$

and associated eigenfunctions, not normalized, are

$$u_{kl}(x) = \sin\left(\frac{\pi k}{a} x_1\right) \sin\left(\frac{\pi l}{b} x_2\right).$$

2. Disk membrane. Set

$$\Omega = \{x \in \mathbb{R}^2 : \ x_1^2 + x_2^2 < R^2\}.$$

In polar coordinates, the eigenvalue problem (4.59), (4.60) is given by

$$\begin{aligned} -\frac{1}{r}\left((r u_r)_r + \frac{1}{r} u_{\theta\theta}\right) &= \lambda u && (4.61)\\ u(R,\theta) &= 0, && (4.62) \end{aligned}$$

here is $u = u(r,\theta) := v(r\cos\theta, r\sin\theta)$. We will find eigenvalues and eigenfunctions by separation of variables

$$u(r,\theta) = v(r)q(\theta),$$

where $v(R) = 0$ and $q(\theta)$ is periodic with period 2π since $u(r,\theta)$ is single valued. This leads to

$$-\frac{1}{r}\left((rv')'q + \frac{1}{r} v q''\right) = \lambda v q.$$

Dividing by vq, provided $vq \neq 0$, we obtain

$$-\frac{1}{r}\left(\frac{(rv'(r))'}{v(r)} + \frac{1}{r}\frac{q''(\theta)}{q(\theta)}\right) = \lambda, \tag{4.63}$$

which implies

$$\frac{q''(\theta)}{q(\theta)} = const. =: -\mu.$$

Thus, we arrive at the eigenvalue problem

$$\begin{aligned} -q''(\theta) &= \mu q(\theta) \\ q(\theta) &= q(\theta + 2\pi). \end{aligned}$$

It follows that eigenvalues μ are real and nonnegative. All solutions of the differential equation are given by

$$q(\theta) = A\sin(\sqrt{\mu}\theta) + B\cos(\sqrt{\mu}\theta),$$

where $A,\ B$ are arbitrary real constants. From the periodicity requirement

$$A\sin(\sqrt{\mu}\theta) + B\cos(\sqrt{\mu}\theta) = A\sin(\sqrt{\mu}(\theta + 2\pi)) + B\cos(\sqrt{\mu}(\theta + 2\pi))$$

it follows[2]

$$\sin(\sqrt{\mu}\pi)\,(A\cos(\sqrt{\mu}\theta + \sqrt{\mu}\pi) - B\sin(\sqrt{\mu}\theta + \sqrt{\mu}\pi)) = 0,$$

which implies, since $A,\ B$ are not zero simultaneously, because we are looking for q not identically zero,

$$\sin(\sqrt{\mu}\pi)\sin(\sqrt{\mu}\theta + \delta) = 0$$

for all θ and a $\delta = \delta(A, B, \mu)$. Consequently the eigenvalues are

$$\mu_n = n^2, \quad n = 0, 1, \ldots\ .$$

Inserting $q''(\theta)/q(\theta) = -n^2$ into (4.63), we obtain the boundary value problem

$$\begin{aligned} r^2v''(r) + rv'(r) + (\lambda r^2 - n^2)v &= 0 \text{ on } (0, R) && (4.64) \\ v(R) &= 0 && (4.65) \\ \sup_{r\in(0,R)} |v(r)| &< \infty. && (4.66) \end{aligned}$$

Set $z = \sqrt{\lambda}r$ and $v(r) = v(z/\sqrt{\lambda}) =: y(z)$, then, see (4.64),

$$z^2y''(z) + zy'(z) + (z^2 - n^2)y(z) = 0,$$

---

[2]

$$\begin{aligned} \sin x - \sin y &= 2\cos\frac{x+y}{2}\sin\frac{x-y}{2} \\ \cos x - \cos y &= -2\sin\frac{x+y}{2}\sin\frac{x-y}{2} \end{aligned}$$

where $z > 0$. Solutions of this differential equations which are bounded at zero are Bessel functions of first kind and n-th order $J_n(z)$. The eigenvalues follows from boundary condition (4.65), i. e., from $J_n(\sqrt{\lambda}R) = 0$. Denote by τ_{nk} the zeros of $J_n(z)$, then the eigenvalues of (4.61)-(4.61) are

$$\lambda_{nk} = \left(\frac{\tau_{nk}}{R}\right)^2$$

and the associated eigenfunctions are

$$\begin{aligned} &J_n(\sqrt{\lambda_{nk}}r)\sin(n\theta), \qquad && n = 1, 2, \ldots \\ &J_n(\sqrt{\lambda_{nk}}r)\cos(n\theta), \qquad && n = 0, 1, 2, \ldots . \end{aligned}$$

Thus the eigenvalues λ_{0k} are simple and $\lambda_{nk},\ n \geq 1$, are double eigenvalues.

Remark. For tables with zeros of $J_n(x)$ and for much more properties of Bessel functions see [25]. One has, in particular, the asymptotic formula

$$J_n(x) = \left(\frac{2}{\pi x}\right)^{1/2} \left(\cos(x - n\pi/2 - \pi/5) + O\left(\frac{1}{x}\right)\right)$$

as $x \to \infty$. It follows from this formula that there are infinitely many zeros of $J_n(x)$.

4.5.3 Inhomogeneous wave equations

Let $\Omega \subset \mathbb{R}^n$ be a bounded and sufficiently regular domain. In this section we consider the initial-boundary value problem

$$\begin{aligned} u_{tt} &= Lu + f(x,t) \ \text{ in } \Omega \times \mathbb{R} && (4.67) \\ u(x,0) &= \phi(x) \ \ x \in \overline{\Omega} && (4.68) \\ u_t(x,0) &= \psi(x) \ \ x \in \overline{\Omega} && (4.69) \\ u(x,t) &= 0 \ \text{ for } x \in \partial\Omega \text{ and } t \in \mathbb{R}^n, && (4.70) \end{aligned}$$

where $u = u(x,t)$, $x = (x_1, \ldots, x_n)$, f, ϕ, ψ are given and L is an elliptic differential operator. Examples for L are:

1. $L = \partial^2/\partial x^2$, oscillating string.

2. $L = \triangle_x$, oscillating membrane.

3.

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a^{ij}(x)u_{x_i}\right),$$

where $a^{ij} = a^{ji}$ are given sufficiently regular functions defined on $\overline{\Omega}$. We assume L is uniformly elliptic, that is, there is a constant $\nu > 0$ such that

$$\sum_{i,j=1}^{n} a^{ij}\zeta_i\zeta_j \ge \nu|\zeta|^2$$

for all $x \in \Omega$ and $\zeta \in \mathbb{R}^n$.

4. Let $u = (u_1, \ldots, u_m)$ and

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(A^{ij}(x)u_{x_i}\right),$$

where $A^{ij} = A^{ji}$ are given sufficiently regular $(m \times m)$-matrices on $\overline{\Omega}$. We assume that L defines an elliptic system. An example for this case is the linear elasticity.

Consider the eigenvalue problem

$$\begin{aligned} -Lv &= \lambda v \ \text{ in } \Omega \qquad (4.71)\\ v &= 0 \ \text{ on } \partial\Omega. \qquad (4.72) \end{aligned}$$

Assume there are infinitely many eigenvalues

$$0 < \lambda_1 \le \lambda_2 \le \ldots \ \to \infty$$

and a system of associated eigenfunctions $v_1,\ v_2, \ldots$ which is complete and orthonormal in $L^2(\Omega)$. This assumption is satisfied if Ω is bounded and if $\partial\Omega$ is sufficiently regular.

For the solution of (4.67)-(4.70) we make the ansatz

$$u(x,t) = \sum_{k=1}^{\infty} v_k(x)w_k(t), \qquad (4.73)$$

with functions $w_k(t)$ which will be determined later. It is assumed that all series are convergent and that following calculations make sense. Let

$$f(x,t) = \sum_{k=1}^{\infty} c_k(t)v_k(x) \qquad (4.74)$$

be Fourier's decomposition of f with respect to the eigenfunctions v_k. We have

$$c_k(t)=\int_\Omega f(x,t)v_k(x)\ dx, \tag{4.75}$$

which follows from (4.74) after multiplying with $v_l(x)$ and integrating over Ω.

Set

$$\langle\phi,v_k\rangle=\int_\Omega \phi(x)v_k(x)\ dx,$$

then

$$\begin{aligned}\phi(x) &= \sum_{k=1}^\infty \langle\phi,v_k\rangle v_k(x)\\ \psi(x) &= \sum_{k=1}^\infty \langle\psi,v_k\rangle v_k(x)\end{aligned}$$

are Fourier's decomposition of ϕ and ψ, respectively.

In the following we will determine $w_k(t)$, which occurs in ansatz (4.73), from the requirement that $u=v_k(x)w_k(t)$ is a solution of

$$u_{tt}=Lu+c_k(t)v_k(x)$$

and that the initial conditions

$$w_k(0)=\langle\phi,v_k\rangle,\quad w_k'(0)=\langle\psi,v_k\rangle$$

are satisfied. From the above differential equation it follows

$$w_k''(t)=-\lambda_k w_k(t)+c_k(t).$$

Thus

$$\begin{aligned}w_k(t) &= a_k\cos(\sqrt{\lambda_k}t)+b_k\sin(\sqrt{\lambda_k}t)\\ &\quad+\frac{1}{\sqrt{\lambda_k}}\int_0^t c_k(\tau)\sin(\sqrt{\lambda_k}(t-\tau))\ d\tau,\end{aligned} \tag{4.76}$$

where

$$a_k=\langle\phi,v_k\rangle,\quad b_k=\frac{1}{\sqrt{\lambda_k}}\langle\psi,v_k\rangle.$$

Summarizing, we have

Proposition 4.2. *The (formal) solution of the initial-boundary value problem (4.67)-(4.70) is given by*

$$u(x,t)=\sum_{k=1}^{\infty} v_k(x)w_k(t),$$

where v_k is a complete orthonormal system of eigenfunctions of (4.71), (4.72) and the functions w_k are defined by (4.76).

The resonance phenomenon

Set in (4.67)-(4.70) $\phi=0$, $\psi=0$ and assume that the external force f is periodic and is given by

$$f(x,t)=A\sin(\omega t)v_n(x),$$

where A, ω are real constants and v_n is one of the eigenfunctions of (4.71), (4.72). It follows

$$c_k(t)=\int_\Omega f(x,t)v_k(x)\ dx=A\delta_{nk}\sin(\omega t).$$

Then the solution of the initial value problem (4.67)-(4.70) is

$$\begin{aligned} u(x,t) &= \frac{Av_n(x)}{\sqrt{\lambda_n}}\int_0^t \sin(\omega\tau)\sin(\sqrt{\lambda_n}(t-\tau))\ d\tau \\ &= Av_n(x)\frac{1}{\omega^2-\lambda_n}\left(\frac{\omega}{\sqrt{\lambda_n}}\sin(\sqrt{\lambda_k}t)-\sin(\omega t)\right), \end{aligned}$$

provided $\omega\neq\sqrt{\lambda_n}$. It follows

$$u(x,t)\to\frac{A}{2\sqrt{\lambda_n}}v_n(x)\left(\frac{\sin(\sqrt{\lambda_n}t)}{\sqrt{\lambda_n}}-t\cos(\sqrt{\lambda_n}t)\right)$$

if $\omega\to\sqrt{\lambda_n}$. The right hand side is also the solution of the initial-boundary value problem if $\omega=\sqrt{\lambda_n}$.

Consequently $|u|$ can be arbitrarily large at some points x and at some times t if $\omega=\sqrt{\lambda_n}$. The frequencies $\sqrt{\lambda_n}$ are called *critical frequencies* at which resonance occurs.

A uniqueness result

The solution of of the initial-boundary value problem (4.67)-(4.70) is unique in the class $C^2(\overline{\Omega}\times\mathbb{R})$.

Proof. Let u_1, u_2 are two solutions, then $u = u_2 - u_1$ satisfies

$$\begin{aligned} u_{tt} &= Lu \ \text{ in } \Omega \times \mathbb{R} \\ u(x,0) &= 0 \ \ x \in \overline{\Omega} \\ u_t(x,0) &= 0 \ \ x \in \overline{\Omega} \\ u(x,t) &= 0 \ \text{ for } x \in \partial\Omega \text{ and } t \in \mathbb{R}^n. \end{aligned}$$

As an example we consider Example 3 from above and set

$$E(t) = \int_\Omega (\sum_{i,j=1}^n a^{ij}(x) u_{x_i} u_{x_j} + u_t u_t)\ dx.$$

Then

$$\begin{aligned} E'(t) &= 2 \int_\Omega (\sum_{i,j=1}^n a^{ij}(x) u_{x_i} u_{x_j t} + u_t u_{tt})\ dx \\ &= 2 \int_{\partial\Omega} (\sum_{i,j=1}^n a^{ij}(x) u_{x_i} u_t n_j)\ dS \\ &\quad +2 \int_\Omega u_t(-Lu + u_t t)\ dx \\ &= 0. \end{aligned}$$

It follows $E(t) = const.$ From $u_t(x,0) = 0$ and $u(x,0) = 0$ we get $E(0) = 0$. Consequently $E(t) = 0$ for all t, which implies, since L is elliptic, that $u(x,t) = const.$ on $\overline{\Omega} \times \mathbb{R}$. Finally, the homogeneous initial and boundary value conditions lead to $u(x,t) = 0$ on $\overline{\Omega} \times \mathbb{R}$. □

4.6 Exercises

1. Show that $u(x,t) \in C^2(\mathbb{R}^2)$ is a solution of the one-dimensional wave equation
$$u_{tt} = c^2 u_{xx}$$
if and only if
$$u(A) + u(C) = u(B) + u(D)$$
holds for all parallelograms $ABCD$ in the (x,t)-plane, which are bounded by characteristic lines, see Figure 4.8.

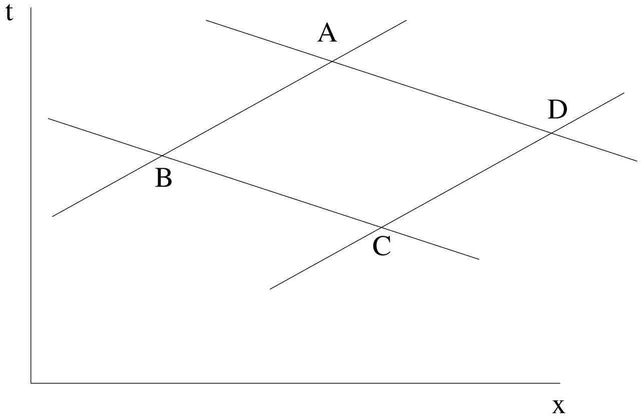

Figure 4.8: Figure to the exercise

2. Method of separation of variables: Let $v_k(x)$ be an eigenfunction to the eigenvalue of the eigenvalue problem $-v''(x) = \lambda v(x)$ in $(0,l)$, $v(0) = v(l) = 0$ and let $w_k(t)$ be a solution of differential equation $-w''(t) = \lambda_k w(t)$. Prove that $v_k(x)w_k(t)$ is a solution of the partial differential equation (wave equation) $u_{tt} = u_{xx}$.

3. Solve for given $f(x)$ and $\mu \in \mathbb{R}$ the initial value problem
$$\begin{array}{rcl} u_t + u_x + \mu u_{xxx} & = & 0 \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+ \\ u(x,0) & = & f(x)\ . \end{array}$$

4. Let $S := \{(x,t);\ t = \gamma x\}$ be spacelike, i. e., $|\gamma| < 1/c^2$) in (x,t)-space, $x = (x_1, x_2, x_3)$. Show that the Cauchy initial value problem $\Box u = 0$

with data for u on S can be transformed using the Lorentz-transform

$$x_1 = \frac{x_1 - \gamma c^2 t}{\sqrt{1-\gamma^2 c^2}},\ x_2' = x_2,\ x_3' = x_3,\ t' = \frac{t - \gamma x_1}{\sqrt{1-\gamma^2 c^2}}$$

into the initial value problem, in new coordinates,

$$\begin{aligned} \Box u &= 0 \\ u(x',0) &= f(x') \\ u_{t'}(x',0) &= g(x')\ . \end{aligned}$$

Here we denote the transformed function by u again.

5. (i) Show that

$$u(x,t) := \sum_{n=1}^{\infty} \alpha_n \cos\left(\frac{\pi n}{l} t\right) \sin\left(\frac{\pi n}{l} x\right)$$

is a C^2-solution of the wave equation $u_{tt} = u_{xx}$ if $|\alpha_n| \le c/n^4$, where the constant c is independent of n.

(ii) Set

$$\alpha_n := \int_0^l f(x) \sin\left(\frac{\pi n}{l} x\right)\ dx.$$

Prove $|\alpha_n| \le c/n^4$, provided $f \in C_0^4(0,l)$.

6. Let Ω be the rectangle $(0,a) \times (0,b)$. Find all eigenvalues and associated eigenfunctions of $-\triangle u = \lambda u$ in Ω, $u = 0$ on $\partial\Omega$.

Hint: Separation of variables.

7. Find a solution of Schrödinger's equation

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\triangle_x \psi + V(x)\psi \quad \text{in } \mathbb{R}^n \times \mathbb{R},$$

which satisfies the side condition

$$\int_{\mathbb{R}}^{n} |\psi(x,t)|^2 dx = 1\ ,$$

provided $E \in \mathbb{R}$ is an (eigenvalue) of the elliptic equation

$$\triangle u + \frac{2m}{\hbar^2}(E - V(x))u = 0 \quad \text{in } \ \mathbb{R}^n$$

under the side condition $\int_{\mathbb{R}}^{n} |u|^2 dx = 1$, $u:\ \mathbb{R}^n \mapsto \mathbb{C}$.
Here is $\psi:\ \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{C}$, $\hbar$ Planck's constant (a small positive constant), $V(x)$ a given potential.

Remark. In the case of a hydrogen atom the potential is $V(x) = -e/|x|$, e is here a positive constant. Then eigenvalues are given by $E_n = -me^4/(2\hbar^2 n^2)$, $n \in \mathbb{N}$, see [22], pp. 202.

8. Find nonzero solutions by using separation of variables of $u_{tt} = \triangle_x u$ in $\Omega \times (0,\infty)$, $u(x,t) = 0$ on $\partial\Omega$, where Ω is the circular cylinder $\Omega = \{(x_1,x_2,x_3) \in \mathbb{R}^n:\ x_1^2 + x_2^2 < R^2,\ 0 < x_3 < h\}$.

9. Solve the initial value problem

$$\begin{aligned} 3u_{tt} - 4u_{xx} &= 0 \\ u(x,0) &= \sin x \\ u_t(x,0) &= 1\ . \end{aligned}$$

10. Solve the initial value problem

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= x^2,\ t > 0,\ x \in \mathbb{R} \\ u(x,0) &= x \\ u_t(x,0) &= 0\ . \end{aligned}$$

Hint: Find a solution of the differential equation independent on t, and transform the above problem into an initial value problem with homogeneous differential equation by using this solution.

11. Find with the method of separation of variables nonzero solutions $u(x,t)$, $0 \le x \le 1,\ 0 \le t < \infty$, of

$$u_{tt} - u_{xx} + u = 0\ ,$$

such that $u(0,t) = 0$, and $u(1,t) = 0$ for all $t \in [0,\infty)$.

12. Find solutions of the equation

$$u_{tt} - c^2 u_{xx} = \lambda^2 u,\ \lambda = const.$$

which can be written as

$$u(x,t) = f(x^2 - c^2t^2) = f(s),\ s := x^2 - c^2t^2$$

with $f(0) = K$, K a constant.

Hint: Transform equation for $f(s)$ by using the substitution $s := z^2/A$ with an appropriate constant A into Bessel's differential equation

$$z^2 f''(z) + zf'(z) + (z^2 - n^2)f = 0,\ z > 0$$

with $n = 0$.

Remark. The above differential equation for u is the transformed telegraph equation (see Section 4.4).

13. Find the formula for the solution of the following Cauchy initial value problem $u_{xy} = f(x, y)$, where S: $y = ax + b$, $a > 0$, and the initial conditions on S are given by

$$\begin{aligned} u &= \alpha x + \beta y + \gamma, \\ u_x &= \alpha, \\ u_y &= \beta, \end{aligned}$$

$a,\ b,\ \alpha,\ \beta,\ \gamma$ constants.

14. Find all eigenvalues μ of

$$\begin{aligned} -q''(\theta) &= \mu q(\theta) \\ q(\theta) &= q(\theta + 2\pi)\ . \end{aligned}$$

Chapter 5

Fourier transform

Fourier's transform is an integral transform which can simplify investigations for linear differential or integral equations since it transforms a differential operator into an algebraic equation.

5.1 Definition, properties

Definition. Let $f \in C_0^s(\mathbb{R}^n)$, $s = 0, 1, \ldots$. The function $\hat{f}$ defined by

$$\widehat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \mathrm{e}^{-i\xi\cdot x} f(x)\ dx, \tag{5.1}$$

where $\xi \in \mathbb{R}^n$, is called *Fourier transform* of f, and the function $\widetilde{g}$ given by

$$\widetilde{g}(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \mathrm{e}^{i\xi\cdot x} g(\xi)\ d\xi \tag{5.2}$$

is called *inverse Fourier transform*, provided the integrals on the right hand side exist.

From (5.1) it follows by integration by parts that differentiation of a function is changed to multiplication of its Fourier transforms, or an analytical operation is converted into an algebraic operation. More precisely, we have

Proposition 5.1.

$$\widehat{D^\alpha f}(\xi) = i^{|\alpha|} \xi^\alpha \widehat{f}(\xi),$$

where $|\alpha| \leq s$.

The following proposition shows that the Fourier transform of f decreases rapidly for $|\xi| \to \infty$, provided $f \in C_0^s(\mathbb{R}^n)$. In particular, the right hand side of (5.2) exists for $g := \hat{f}$ if $f \in C_0^{n+1}(\mathbb{R}^n)$.

Proposition 5.2. Assume $g \in C_0^s(\mathbb{R}^n)$, then there is a constant $M = M(n,s,g)$ such that

$$|\widehat{g}(\xi)| \le \frac{M}{(1+|\xi|)^s}.$$

Proof. Let $\xi = (\xi_1, \ldots, \xi_n)$ be fixed and let j be an index such that $|\xi_j| = \max_k |\xi_k|$. Then

$$|\xi| = \left(\sum_{k=1}^n \xi_k^2\right)^{1/2} \le \sqrt{n}|\xi_j|$$

which implies

$$\begin{aligned}(1+|\xi|)^s &= \sum_{k=0}^s \binom{s}{k} |\xi|^k \\ &\le 2^s \sum_{k=0}^s n^{k/2} |\xi_j|^k \\ &\le 2^s n^{s/2} \sum_{|\alpha| \le s} |\xi^\alpha|.\end{aligned}$$

This inequality and Proposition 5.1 imply

$$\begin{aligned}(1+|\xi|)^s |\widehat{g}(\xi)| &\le 2^s n^{s/2} \sum_{|\alpha| \le s} |(i\xi)^\alpha \widehat{g}(\xi)| \\ &\le 2^s n^{s/2} \sum_{|\alpha| \le s} \int_{\mathbb{R}^n} |D^\alpha g(x)| \; dx =: M.\end{aligned}$$

□

The notation *inverse* Fourier transform for (5.2) is justified by

Theorem 5.1. $\widetilde{\widehat{f}} = f$ *and* $\widehat{\widetilde{f}} = f$.

Proof. See [27], for example. We will prove the first assertion

$$(2\pi)^{-n/2} \int_{\mathbb{R}^n} \mathrm{e}^{i\xi \cdot x} \widehat{f}(\xi) \; d\xi = f(x) \tag{5.3}$$

here. The proof of the other relation is left as an exercise. All integrals appearing in the following exist, see Proposition 5.2 and the special choice of g.

(i) Formula

$$\int_{\mathbb{R}^n} g(\xi)\widehat{f}(\xi)\mathrm{e}^{ix\cdot\xi}\ d\xi = \int_{\mathbb{R}^n} \widehat{g}(y)f(x+y)\ dy \tag{5.4}$$

follows by direct calculation:

$$\begin{aligned}
&\int_{\mathbb{R}^n} g(\xi)\left((2\pi)^{-n/2}\int_{\mathbb{R}^n} \mathrm{e}^{-ix\cdot y}f(y)\ dy\right)\mathrm{e}^{ix\cdot\xi}\ d\xi \\
&\quad = (2\pi)^{-n/2}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} g(\xi)\mathrm{e}^{-i\xi\cdot(y-x)}\ d\xi\right)f(y)\ dy \\
&\quad = \int_{\mathbb{R}^n} \widehat{g}(y-x)f(y)\ dy \\
&\quad = \int_{\mathbb{R}^n} \widehat{g}(y)f(x+y)\ dy.
\end{aligned}$$

(ii) Formula

$$(2\pi)^{-n/2}\int_{\mathbb{R}^n} \mathrm{e}^{-iy\cdot\xi}g(\varepsilon\xi)\ d\xi = \varepsilon^{-n}\widehat{g}(y/\varepsilon) \tag{5.5}$$

for each $\varepsilon > 0$ follows after substitution $z = \varepsilon\xi$ in the left hand side of (5.1).

(iii) Equation

$$\int_{\mathbb{R}^n} g(\varepsilon\xi)\widehat{f}(\xi)\mathrm{e}^{ix\cdot\xi}\ d\xi = \int_{\mathbb{R}^n} \widehat{g}(y)f(x+\varepsilon y)\ dy \tag{5.6}$$

follows from (5.4) and (5.5). Set $G(\xi) := g(\varepsilon\xi)$, then (5.4) implies

$$\int_{\mathbb{R}^n} G(\xi)\widehat{f}(\xi)\mathrm{e}^{ix\cdot\xi}\ d\xi = \int_{\mathbb{R}^n} \widehat{G}(y)f(x+y)\ dy.$$

Since, see (5.5),

$$\begin{aligned}
\widehat{G}(y) &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-iy\cdot\xi}g(\varepsilon\xi)\ d\xi \\
&= \varepsilon^{-n}\widehat{g}(y/\varepsilon),
\end{aligned}$$

we arrive at

$$\begin{aligned}\int_{\mathbb{R}^n} g(\varepsilon\xi)\widehat{f}(\xi) &= \int_{\mathbb{R}^n} \varepsilon^{-n}\widehat{g}(y/\varepsilon)f(x+y)\ dy\\ &= \int_{\mathbb{R}^n} \widehat{g}(z)f(x+\varepsilon z)\ dz.\end{aligned}$$

Letting $\varepsilon \to 0$, we get

$$g(0)\int_{\mathbb{R}^n} \widehat{f}(\xi)e^{ix\cdot\xi}\ d\xi = f(x)\int_{\mathbb{R}^n} \widehat{g}(y)\ dy. \tag{5.7}$$

Set

$$g(x) := \mathrm{e}^{-|x|^2/2},$$

then

$$\int_{\mathbb{R}^n} \widehat{g}(y)\ dy = (2\pi)^{n/2}. \tag{5.8}$$

Since $g(0) = 1$, the first assertion of Theorem 5.1 follows from (5.7) and (5.8). It remains to show (5.8).

(iv) *Proof of (5.8).* We will show

$$\begin{aligned}\widehat{g}(y) : &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} \mathrm{e}^{-|x|^2/2}\mathrm{e}^{-ix\cdot x}\ dx\\ &= \mathrm{e}^{-|y|^2/2}.\end{aligned}$$

The proof of

$$\int_{\mathbb{R}^n} \mathrm{e}^{-|y|^2/2}\ dy = (2\pi)^{n/2}$$

is left as an exercise. Since

$$-\left(\frac{x}{\sqrt{2}} + i\frac{y}{\sqrt{2}}\right)\cdot\left(\frac{x}{\sqrt{2}} + i\frac{y}{\sqrt{2}}\right) = -\left(\frac{|x|^2}{2} + ix\cdot y - \frac{|y|^2}{2}\right)$$

it follows

$$\begin{aligned}\int_{\mathbb{R}^n} \mathrm{e}^{-|x|^2/2}\mathrm{e}^{-ix\cdot y}\ dx &= \int_{\mathbb{R}^n} \mathrm{e}^{-\eta^2}\mathrm{e}^{-|y|^2/2}\ dx\\ &= \mathrm{e}^{-|y|^2/2}\int_{\mathbb{R}^n} \mathrm{e}^{-\eta^2}\ dx\\ &= 2^{n/2}\mathrm{e}^{-|y|^2/2}\int_{\mathbb{R}^n} \mathrm{e}^{-\eta^2}\ d\eta\end{aligned}$$

where

$$\eta := \frac{x}{\sqrt{2}} + i\frac{y}{\sqrt{2}}.$$

Consider first the one-dimensional case. According to Cauchy's theorem we have

$$\oint_C \mathrm{e}^{-\eta^2}\ d\eta = 0,$$

where the integration is along the curve C which is the union of four curves as indicated in Figure 5.1.

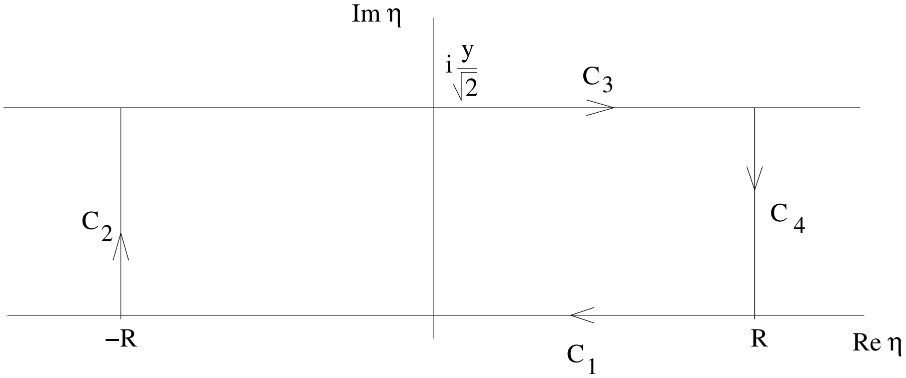

Figure 5.1: Proof of (5.8)

Consequently

$$\int_{C_3} \mathrm{e}^{-\eta^2}\ d\eta = \frac{1}{\sqrt{2}}\int_{-R}^{R} \mathrm{e}^{-x^2/2}\ dx - \int_{C_2} \mathrm{e}^{-\eta^2}\ d\eta - \int_{C_4} \mathrm{e}^{-\eta^2}\ d\eta.$$

It follows

$$\lim_{R\to\infty}\int_{C_3} \mathrm{e}^{-\eta^2}\ d\eta = \sqrt{\pi}$$

since

$$\lim_{R\to\infty}\int_{C_k} \mathrm{e}^{-\eta^2}\ d\eta = 0,\quad k = 2,\ 4.$$

The case $n > 1$ can be reduced to the one-dimensional case as follows. Set

$$\eta = \frac{x}{\sqrt{2}} + i\frac{y}{\sqrt{2}} = (\eta_1, \ldots, \eta_n),$$

where

$$\eta_l = \frac{x_l}{\sqrt{2}} + i\frac{y_l}{\sqrt{2}}.$$

From $d\eta = d\eta_1 \dots d\eta_l$ and

$$\mathrm{e}^{-\eta^2} = \mathrm{e}^{-\sum_{l=1}^n \eta_l^2} = \prod_{l=1}^n \mathrm{e}^{-\eta_l^2}$$

it follows

$$\int_{\mathbb{R}^n} e^{-\eta^2}\ d\eta = \prod_{l=1}^n \int_{\Gamma_l} \mathrm{e}^{-\eta_l^2}\ d\eta_l,$$

where for fixed y

$$\Gamma_l = \{z \in \mathbb{C} :\ z = \frac{x_l}{\sqrt{2}} + i\frac{y_l}{\sqrt{2}}, -\infty < x_l < +\infty\}.$$

□

There is a useful class of functions for which the integrals in the definition of $\widehat{f}$ and $\widetilde{f}$ exist.

For $u \in C^\infty(\mathbb{R}^n)$ we set

$$q_{j,k}(u) := \max_{\alpha:\ |\alpha| \le k} \left(\sup_{\mathbb{R}^n} \left((1+|x|^2)^{j/2} |D^\alpha u(x)| \right) \right).$$

Definition. The *Schwartz class* of rapidly degreasing functions is

$$\mathcal{S}(\mathbb{R}^n) = \{u \in C^\infty(\mathbb{R}^n) :\ q_{j,k}(u) < \infty \text{ for any } j,k \in \mathbb{N} \cup \{0\}\}.$$

This space is a Frechét space.

Proposition 5.3. *Assume $u \in \mathcal{S}(\mathbb{R}^n)$, then $\widehat{u}$ and $\widetilde{u} \in \mathcal{S}(\mathbb{R}^n)$.*

Proof. See [24], Chapter 1.2, for example, or an exercise.

5.1.1 Pseudodifferential operators

The properties of Fourier transform lead to a general theory for linear partial differential or integral equations. In this subsection we define

$$D_k = \frac{1}{i}\frac{\partial}{\partial x_k}, \quad k = 1, \dots, n,$$

and for each multi-index α as in Subsection 3.5.1

$$D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}.$$

Thus

$$D^\alpha = \frac{1}{i^{|\alpha|}} \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}.$$

Let

$$p(x, D) := \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha,$$

be a linear partial differential of order m, where a_α are given sufficiently regular functions.

According to Theorem 5.1 and Proposition 5.3, we have, at least for $u \in \mathcal{S}(\mathbb{R}^n)$,

$$u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \mathrm{e}^{ix\cdot\xi} \widehat{u}(\xi)\ d\xi,$$

which implies

$$D^\alpha u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \mathrm{e}^{ix\cdot\xi} \xi^\alpha \widehat{u}(\xi)\ d\xi.$$

Consequently

$$p(x, D)u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \mathrm{e}^{ix\cdot\xi} p(x, \xi) \widehat{u}(\xi)\ d\xi, \tag{5.9}$$

where

$$p(x, \xi) = \sum_{|\alpha| \le m} a_\alpha(x) \xi^\alpha.$$

The right hand side of (5.9) makes sense also for more general functions $p(x, \xi)$, not only for polynomials.

Definition. The function $p(x, \xi)$ is called *symbol* and

$$(Pu)(x) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} \mathrm{e}^{ix\cdot\xi} p(x, \xi) \widehat{u}(\xi)\ d\xi$$

is said to be *pseudodifferential operator*.

An important class of symbols for which the right hand side in this definition of a pseudodifferential operator is defined is S^m which is the subset of $p(x, \xi) \in C^\infty(\Omega \times \mathbb{R}^n)$ such that

$$|D_x^\beta D_\xi^\alpha p(x, \xi)| \le C_{K,\alpha,\beta}(p)\, (1 + |\xi|)^{m - |\alpha|}$$

for each compact $K \subset \Omega$.

Above we have seen that linear differential operators define a class of pseudodifferential operators. Even integral operators can be written (formally) as pseudodifferential operators. Let

$$(Pu)(x) = \int_{\mathbb{R}^n} K(x,y)u(y)\ dy$$

be an integral operator. Then

$$\begin{aligned}(Pu)(x) &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} K(x,y)\int_{\mathbb{R}^n} \mathrm{e}^{ix\cdot\xi}\xi^{\alpha}\widehat{u}(\xi)\ d\xi \\ &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} \mathrm{e}^{ix\cdot\xi}\left(\int_{\mathbb{R}^n} \mathrm{e}^{i(y-x)\cdot\xi}K(x,y)\ dy\right)\widehat{u}(\xi).\end{aligned}$$

Then the symbol associated to the above integral operator is

$$p(x,\xi) = \int_{\mathbb{R}^n} \mathrm{e}^{i(y-x)\cdot\xi}K(x,y)\ dy.$$

5.2 Exercises

1. Show
$$\int_{\mathbb{R}^n} \mathrm{e}^{-|y|^2/2}\, dy = (2\pi)^{n/2}.$$
2. Show that $u \in \mathcal{S}(\mathbb{R}^n)$ implies $\hat{u},\ \widetilde{u} \in \mathcal{S}(\mathbb{R}^n)$.
3. Give examples for functions $p(x,\xi)$ which satisfy $p(x,\xi) \in S^m$.
4. Find a formal solution of Cauchy's initial value problem for the wave equation by using Fourier's transform.

Chapter 6

Parabolic equations

Here we consider linear parabolic equations of second order. An example is the heat equation

$$u_t = a^2 \triangle u,$$

where $u = u(x,t)$, $x \in \mathbb{R}^3$, $t \geq 0$, and a^2 is a positive constant called conductivity coefficient. The heat equation has its origin in physics where $u(x,t)$ is the temperature at x at time t, see [20], p. 394, for example.

Remark 1. After scaling of axis we can assume $a = 1$.

Remark 2. By setting $t := -t$, the heat equation changes to an equation which is called backward equation. This is the reason for the fact that the heat equation describes irreversible processes in contrast to the wave equation $\Box u = 0$ which is invariant with respect the mapping $t \mapsto -t$. Mathematically, it means that it is not possible, in general, to find the distribution of temperature at an earlier time $t < t_0$ if the distribution is given at t_0.

Consider the initial value problem for $u = u(x,t)$, $u \in C^\infty(\mathbb{R}^n \times R_+)$,

$$\begin{aligned} u_t &= \triangle u \ \text{ in } x \in \mathbb{R}^n,\ t \geq 0, && (6.1) \\ u(x,0) &= \phi(x), && (6.2) \end{aligned}$$

where $\phi \in C(\mathbb{R}^n)$ is given and $\triangle \equiv \triangle_x$.

6.1 Poisson's formula

Assume u is a solution of (6.1), then, since Fourier transform is a linear mapping,

$$\widehat{u_t - \triangle u} = \hat{0}.$$

From properties of the Fourier transform, see Proposition 5.1, we have

$$\widehat{\triangle u} = \sum_{k=1}^{n} \widehat{\frac{\partial^2 u}{\partial x_k^2}} = \sum_{k=1}^{n} i^2 \xi_k^2 \widehat{u}(\xi),$$

provided the transforms exist. Thus we arrive at the ordinary differential equation for the Fourier transform of u

$$\frac{d\widehat{u}}{dt} + |\xi|^2 \widehat{u} = 0,$$

where ξ is considered as a parameter. The solution is

$$\widehat{u}(\xi, t) = \widehat{\phi}(\xi) \mathrm{e}^{-|\xi|^2 t}$$

since $\widehat{u}(\xi, 0) = \widehat{\phi}(\xi)$. From Theorem 5.1 it follows

$$\begin{aligned} u(x,t) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \widehat{\phi}(\xi) \mathrm{e}^{-|\xi|^2 t} \mathrm{e}^{i\xi\cdot x} \, d\xi \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} \phi(y) \left(\int_{\mathbb{R}^n} \mathrm{e}^{i\xi\cdot(x-y)-|\xi|^2 t} \, d\xi \right) dy. \end{aligned}$$

Set

$$K(x,y,t) = (2\pi)^{-n} \int_{\mathbb{R}^n} \mathrm{e}^{i\xi\cdot(x-y)-|\xi|^2 t} \, d\xi.$$

By the same calculations as in the proof of Theorem 5.1, step (vi), we find

$$K(x,y,t) = (4\pi t)^{-n/2} \mathrm{e}^{-|x-y|^2/4t}. \tag{6.3}$$

Thus we have

$$u(x,t) = \frac{1}{\left(2\sqrt{\pi t}\right)^n} \int_{\mathbb{R}^n} \phi(z) \mathrm{e}^{-|x-z|^2/4t} \, dz. \tag{6.4}$$

Definition. Formula (6.4) is called *Poisson's formula* and the function K defined by (6.3) *heat kernel* or *fundamental solution* of the heat equation.

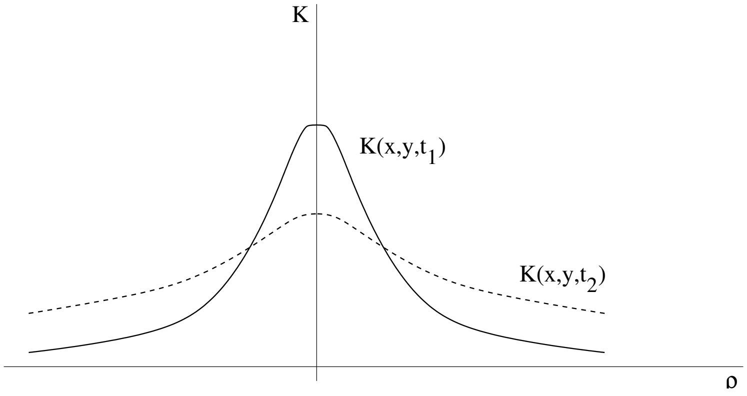

Figure 6.1: Kernel $K(x,y,t)$, $\rho = |x-y|$, $t_1 < t_2$

Proposition 6.1 *The kernel* K *has following properties:*

(i) $K(x,y,t) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+)$,

(ii) $(\partial/\partial t \ - \triangle)K(x,y,t) = 0,\ t > 0$,

(iii) $K(x,y,t) > 0,\ t > 0$,

(iv) $\int_{\mathbb{R}^n}\ K(x,y,t)\ dy = 1,\ x \in \mathbb{R}^n,\ t > 0$,

(v) For each fixed $\delta > 0$*:*

$$\lim_{\substack{t \to 0\\ t > 0}} \int_{\mathbb{R}^n \setminus B_\delta(x)} K(x,y,t)\ dy = 0$$

uniformly for $x \in \mathbb{R}^n$.

Proof. (i) and (iii) are obviously, and (ii) follows from the definition of K. Equations (iv) and (v) hold since

$$\begin{aligned}\int_{\mathbb{R}^n \setminus B_\delta(x)} K(x,y,t)\ dy &= \int_{\mathbb{R}^n \setminus B_\delta(x)} (4\pi t)^{-n/2} \mathrm{e}^{-|x-y|^2/4t}\ dy \\ &= \pi^{-n/2} \int_{\mathbb{R}^n \setminus B_{\delta/\sqrt{4t}}(0)} e^{-|\eta|^2}\ d\eta\end{aligned}$$

by using the substitution $y = x + (4t)^{1/2}\eta$. For fixed $\delta > 0$ it follows (v) and for $\delta := 0$ we obtain (iv). □

Theorem 6.1. *Assume $\phi \in C(\mathbb{R}^n)$ and $\sup_{\mathbb{R}^n} |\phi(x)| < \infty$. Then $u(x,t)$ given by Poisson's formula (6.4) is in $C^\infty(\mathbb{R}^n \times \mathbb{R}_+)$, continuous on $\mathbb{R}^n \times [0,\infty)$ and a solution of the initial value problem (6.1), (6.2).*

Proof. It remains to show

$$\lim_{\substack{x \to \xi \\ t \to 0}} u(x,t) = \phi(\xi).$$

Since ϕ is continuous there exists for given $\varepsilon > 0$ a $\delta = \delta(\varepsilon)$ such that $|\phi(y) - \phi(\xi)| < \varepsilon$ if $|y - \xi| < 2\delta$. Set $M := \sup_{\mathbb{R}^n} |\phi(y)|$. Then, see Proposition 6.1,

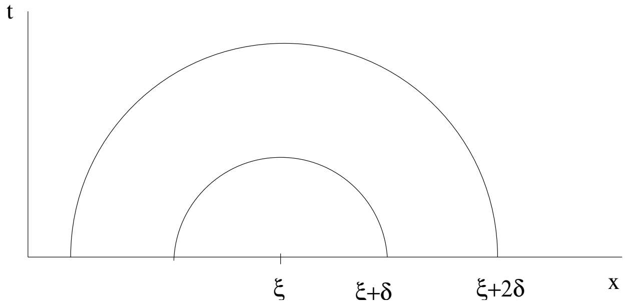

Figure 6.2: Figure to the proof of Theorem 6.1

$$u(x,t) - \phi(\xi) = \int_{\mathbb{R}^n} K(x,y,t)\,(\phi(y) - \phi(\xi))\;dy.$$

It follows, if $|x-\xi|<\delta$ and $t>0$, that

$$\begin{aligned}
|u(x,t)-\phi(\xi)| &\le \int_{B_\delta(x)} K(x,y,t)\,|\phi(y)-\phi(\xi)|\ dy \\
&\quad + \int_{\mathbb{R}^n\setminus B_\delta(x)} K(x,y,t)\,|\phi(y)-\phi(\xi)|\ dy \\
&\le \int_{B_{2\delta}(x)} K(x,y,t)\,|\phi(y)-\phi(\xi)|\ dy \\
&\quad +2M\int_{\mathbb{R}^n\setminus B_\delta(x)} K(x,y,t)\ dy \\
&\le \varepsilon\int_{\mathbb{R}^n} K(x,y,t)\ dy+2M\int_{\mathbb{R}^n\setminus B_\delta(x)} K(x,y,t)\ dy \\
&< 2\varepsilon
\end{aligned}$$

if $0<t\le t_0$, t_0 sufficiently small. □

Remarks. 1. Uniqueness follows under the additional growth assumption

$$|u(x,t)|\le M\mathrm{e}^{a|x|^2}\ \text{ in } D_T,$$

where M and a are positive constants, see Proposition 6.2 below.
In the one-dimensional case, one has uniqueness in the class $u(x,t)\ge 0$ in D_T, see [10], pp. 222.

2. $u(x,t)$ defined by Poisson's formula depends on all values $\phi(y)$, $y\in\mathbb{R}^n$. That means, a perturbation of ϕ, even far from a fixed x, has influence to the value $u(x,t)$. In physical terms, this means that heat travels with infinite speed, in contrast to the experience.

6.2 Inhomogeneous heat equation

Here we consider the initial value problem for $u=u(x,t)$, $u\in C^\infty(\mathbb{R}^n\times R_+)$,

$$\begin{aligned}
u_t-\triangle u &= f(x,t)\ \text{ in } x\in\mathbb{R}^n,\ t\ge 0,\\
u(x,0) &= \phi(x),
\end{aligned}$$

where ϕ and f are given. From

$$\widehat{u_t-\triangle u}=\widehat{f(x,t)}$$

we obtain an initial value problem for an ordinary differential equation:

$$\begin{aligned} \frac{d\widehat{u}}{dt}+|\xi|^2\widehat{u} &= \widehat{f}(\xi,t) \\ \widehat{u}(\xi,0) &= \widehat{\phi}(\xi). \end{aligned}$$

The solution is given by

$$\widehat{u}(\xi,t)=\mathrm{e}^{-|\xi|^2t}\widehat{\phi}(\xi)+\int_0^t\ \mathrm{e}^{-|\xi|^2(t-\tau)}\widehat{f}(\xi,\tau)\ d\tau.$$

Applying the inverse Fourier transform and a calculation as in the proof of Theorem 5.1, step (vi), we get

$$\begin{aligned} u(x,t) &= (2\pi)^{-n/2}\int_{\mathbb{R}^n}\ \mathrm{e}^{ix\cdot\xi}\Big(\mathrm{e}^{-|\xi|^2t}\widehat{\phi}(\xi) \\ &\qquad +\int_0^t\ \mathrm{e}^{-|\xi|^2(t-\tau)}\widehat{f}(\xi,\tau)\ d\tau\Big)\ d\xi. \end{aligned}$$

From the above calculation for the homogeneous problem and calculation as in the proof of Theorem 5.1, step (vi), we obtain the formula

$$\begin{aligned} u(x,t) &= \frac{1}{(2\sqrt{\pi t})^n}\int_{\mathbb{R}^n}\ \phi(y)\mathrm{e}^{-|y-x|^2/(4t)}\ dy \\ &\quad +\int_0^t\int_{\mathbb{R}^n}\ f(y,\tau)\frac{1}{\Big(2\sqrt{\pi(t-\tau)}\Big)^n}\ \mathrm{e}^{-|y-x|^2/(4(t-\tau))}\ dy\ d\tau. \end{aligned}$$

This function $u(x,t)$ is a solution of the above inhomogeneous initial value problem provided

$$\phi\in C(\mathbb{R}^n),\quad \sup_{\mathbb{R}^n}|\phi(x)|<\infty$$

and if

$$f\in C(\mathbb{R}^n\times[0,\infty)),\quad M(\tau):=\sup_{\mathbb{R}^n}|f(y,\tau)|<\infty,\ 0\le\tau<\infty.$$

6.3 Maximum principle

Let $\Omega\subset\mathbb{R}^n$ be a *bounded* domain. Set

$$\begin{aligned} D_T &= \Omega\times(0,T),\quad T>0, \\ S_T &= \{(x,t):\ (x,t)\in\Omega\times\{0\}\text{ or }(x,t)\in\partial\Omega\times[0,T]\}, \end{aligned}$$

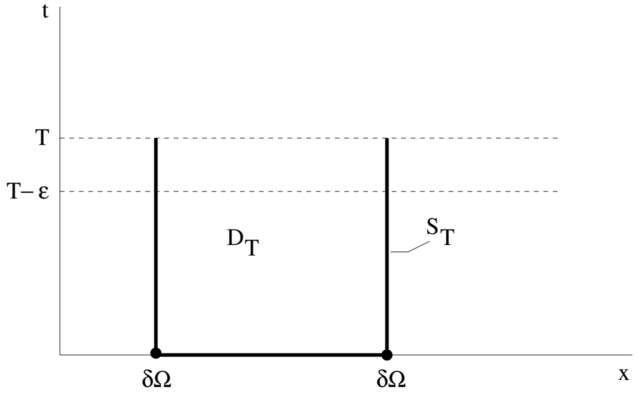

Figure 6.3: Notations to the maximum principle

see Figure 6.3

Theorem 6.2. *Assume* $u \in C(\overline{D_T})$, *that* u_t, $u_{x_ix_k}$ *exist and are continuous in* D_T, *and*

$$u_t - \triangle u \le 0 \quad \text{in } D_T.$$

Then

$$\max_{\overline{D_T}} u(x,t) = \max_{S_T} u.$$

Proof. Assume initially $u_t - \triangle u < 0$ in D_T. Let $\varepsilon > 0$ be small and $0 < \varepsilon < T$. Since $u \in C(\overline{D_{T-\varepsilon}})$, there is an $(x_0, t_0) \in \overline{D_{T-\varepsilon}}$ such that

$$u(x_0, t_0) = \max_{\overline{D_{T-\varepsilon}}} u(x,t).$$

Case (i). Let $(x_0, t_0) \in D_{T-\varepsilon}$. Hence, since $D_{T-\varepsilon}$ is open, $u_t(x_0, t_0) = 0$, $u_{x_l}(x_0, t_0) = 0$, $l = 1, \ldots, n$ and

$$\sum_{l,k=1}^{n} u_{x_lx_k}(x_0, t_0)\zeta_l\zeta_k \le 0 \quad \text{for all } \zeta \in \mathbb{R}^n.$$

The previous inequality implies that $u_{x_kx_k}(x_0, t_0) \le 0$ for each k. Thus we arrived at a contradiction to $u_t - \triangle u < 0$ in D_T.

Case (ii). Assume $(x_0,t_0)\in\Omega\times\{T-\varepsilon\}$. Then it follows as above $\triangle u\le 0$ in (x_0,t_0), and from $u(x_0,t_0)\ge u(x_0,t)$, $t\le t_0$, one concludes that $u_t(x_0,t_0)\ge 0$. We arrived at a contradiction to $u_t-\triangle u<0$ in D_T again.

Summarizing, we have shown that

$$\max_{\overline{D_{T-\varepsilon}}} u(x,t)=\max_{T-\varepsilon} u(x,t).$$

Thus there is an $(x_\varepsilon,t_\varepsilon)\in S_{T-\varepsilon}$ such that

$$u(x_\varepsilon,t_\varepsilon)=\max_{\overline{D_{T-\varepsilon}}} u(x,t).$$

Since u is continuous on $\overline{D}_T$, we have

$$\lim_{\varepsilon\to 0}\max_{\overline{D_{T-\varepsilon}}} u(x,t)=\max_{\overline{D_T}} u(x,t).$$

It follows that there is $(\overline{x},\overline{t})\in S_T$ such that

$$u(\overline{x},\overline{t})=\max_{\overline{D_T}} u(x,t)$$

since $S_{T-\varepsilon}\subset S_T$ and S_T is compact. Thus, theorem is shown under the assumption $u_t-\triangle u<0$ in D_T. Now assume $u_t-\triangle u\le 0$ in D_T. Set

$$v(x,t):=u(x,t)-kt,$$

where k is a positive constant. Then

$$v_t-\triangle v=u_t-\triangle u-k<0.$$

From above we have

$$\begin{aligned}\max_{\overline{D_T}} u(x,t) &= \max_{\overline{D_T}}(v(x,t)+kt)\\ &\le \max_{\overline{D_T}} v(x,t)+kT\\ &= \max_{S_T} v(x,t)+kT\\ &\le \max_{S_T} u(x,t)+kT,\end{aligned}$$

Letting $k\to 0$, we obtain

$$\max_{\overline{D_T}} u(x,t)\le\max_{S_T} u(x,t).$$

Since $S_T \subset \overline{D_T}$, the theorem is shown. □

If we replace in the above theorem the bounded domain Ω by $\mathbb{R}^n$, then the result remains true provided we assume an *additional* growth assumption for u. More precisely, we have the following result which is a corollary of the previous theorem. Set for a fixed T, $0 < T < \infty$,

$$D_T = \{(x,t):\ x \in \mathbb{R}^n,\ 0 < t < T\}.$$

Proposition 6.2. *Assume $u \in C(\overline{D_T})$, that u_t, $u_{x_i x_k}$ exist and are continuous in D_T,*

$$u_t - \triangle u \le 0 \ \ \text{in } D_T,$$

and additionally that u satisfies the growth condition

$$u(x,t) \le M e^{a|x|^2},$$

where M and a are positive constants. Then

$$\max_{\overline{D_T}} u(x,t) = \max_{S_T} u.$$

It follows immediately the

Corollary. *The initial value problem $u_t - \triangle u = 0$ in D_T, $u(x,0) = f(x)$, $x \in \mathbb{R}^n$, has a unique solution in the class defined by $u \in C(\overline{D_T})$, u_t, $u_{x_i x_k}$ exist and are continuous in D_T and $|u(x,t)| \le M e^{a|x|^2}$.*

Proof of Proposition 6.2. See [10], pp. 217. We can assume that $4aT < 1$, since the *finite* interval can be divided into finite many intervals of equal length τ with $4a\tau < 1$. Then we conclude successively for k that

$$u(x,t) \le \sup_{y \in \mathbb{R}^n} u(y, k\tau) \le \sup_{y \in \mathbb{R}^n} u(y,0)$$

for $k\tau \le t \le (k+1)\tau$, $k = 0, \ldots, N-1$, where $N = T/\tau$.

There is an $\epsilon > 0$ such that $4a(T+\epsilon) < 1$. Consider the comparison function

$$\begin{aligned} v_\mu(x,t) :&= u(x,t) - \mu\,(4\pi(T+\epsilon-t))^{-n/2}\, \mathrm{e}^{|x-y|^2/(4(T+\epsilon-t))} \\ &= u(x,t) - \mu K(ix, iy, T+\epsilon-t) \end{aligned}$$

for fixed $y \in \mathbb{R}^n$ and for a constant $\mu > 0$. Since the heat kernel $K(ix, iy, t)$ satisfies $K_t = \triangle K_x$, we obtain

$$\frac{\partial}{\partial t} v_\mu - \triangle v_\mu = u_t - \triangle u \le 0.$$

Set for a constant $\rho > 0$

$$D_{T,\rho} = \{(x,t) :\ |x-y| < \rho,\ 0 < t < T\}.$$

Then we obtain from Theorem 6.2 that

$$v_\mu(y,t) \le \max_{S_{T,\rho}} v_\mu,$$

where $S_{T,\rho} \equiv S_T$ of Theorem 6.2 with $\Omega = B_\rho(y)$, see Figure 6.3. On the bottom of $S_{T,\rho}$ we have, since $\mu K > 0$,

$$v_\mu(x,0) \le u(x,0) \le \sup_{z \in \mathbb{R}^n} f(z).$$

On the cylinder part $|x-y| = \rho$, $0 \le t \le T$, of $S_{T,\rho}$ it is

$$\begin{aligned} v_\mu(x,t) &\le Me^{a|x|^2} - \mu \left(4\pi(T+\epsilon-t)\right)^{-n/2} e^{\rho^2/(4(T+\epsilon-t))} \\ &\le Me^{a(|y|+\rho)^2} - \mu \left(4\pi(T+\epsilon)\right)^{-n/2} e^{\rho^2/(4(T+\epsilon))} \\ &\le \sup_{z \in \mathbb{R}^n} f(z) \end{aligned}$$

for all $\rho > \rho_0(\mu)$, ρ_0 sufficiently large. We recall that $4a(T+\epsilon) < 1$. Summarizing, we have

$$\max_{S_{T,\rho}} v_\mu(x,t) \le \sup_{z \in \mathbb{R}^n} f(z)$$

if $\rho > \rho_0(\mu)$. Thus

$$v_\mu(y,t) \le \max_{S_{T,\rho}} v_\mu(x,t) \le \sup_{z \in \mathbb{R}^n} f(z)$$

if $\rho > \rho_0(\mu)$. Since

$$v_\mu(y,t) = u(y,t) - \mu \left(4\pi(T+\epsilon-t)\right)^{-n/2}$$

it follows

$$u(y,t) - \mu \left(4\pi(T+\epsilon-t)\right)^{-n/2} \le \sup_{z \in \mathbb{R}^n} f(z).$$

Letting $\mu \to 0$, we obtain the assertion of the proposition. □

The above maximum principle of Theorem 6.2 holds for a large class of parabolic differential operators, even for degenerate equations. Set

$$Lu = \sum_{i,j=1}^{n} a^{ij}(x,t)u_{x_i x_j},$$

where $a^{ij} \in C(D_T)$ are real, $a^{ij} = a^{ji}$, and the matrix (a^{ij}) is nonnegative, that is,

$$\sum_{i,j=1}^{n} a^{ij}(x,t)\zeta_i\zeta_j \ge 0 \ \text{ for all } \zeta \in \mathbb{R}^n,$$

and $(x,t) \in D_T$.

Theorem 6.3. *Assume $u \in C(\overline{D_T})$, that u_t, $u_{x_i x_k}$ exist and are continuous in D_T, and*

$$u_t - Lu \le 0 \quad \textit{in } D_T.$$

Then

$$\max_{\overline{D_T}} u(x,t) = \max_{S_T} u.$$

Proof. (i) One proof is a consequence of the following lemma: Let A, B real, symmetric and nonnegative matrices. Nonnegative means that all eigenvalues are nonnegative. Then trace $(AB) \equiv \sum_{i,j=1}^{n} a^{ij} b_{ij} \ge 0$, see an exercise.

(ii) Another proof exploits transform to principle axis directly: Let $U = (z_1, \ldots, z_n)$, where z_l is an orthonormal system of eigenvectors to the eigenvalues λ_l of the matrix $A = (a^{i,j}(x_0,t_0))$. Set $\zeta = U\eta$, $x = U^T(x - x_0)y$ and $v(y) = u(x_0 + Uy, t_0)$, then

$$\begin{aligned} 0 &\le \sum_{i,j=1}^{n} a^{ij}(x_0,t_0)\zeta_i\zeta_j = \sum_{i=1}^{n} \lambda_i \eta_i^2 \\ 0 &\ge \sum_{i,j=1}^{n} u_{x_i x_j}\zeta_i\zeta_j = \sum_{i=1}^{n} v_{y_i y_i}\eta_i^2. \end{aligned}$$

It follows $\lambda_i \ge 0$ and $v_{y_i y_i} \le 0$ for all i. Consequently

$$\sum_{i,j=1}^{n} a^{ij}(x_0,t_0)u_{x_i x_j}(x_0,t_0) = \sum_{i=1}^{n} \lambda_i v_{y_i y_i} \le 0.$$

□

6.4 Initial-boundary value problem

Consider the initial-boundary value problem for $c = c(x,t)$

$$\begin{aligned} c_t &= D\triangle c \;\text{ in } \Omega\times(0,\infty) \quad (6.5)\\ c(x,0) &= c_0(x) \;\; x\in\overline{\Omega} \quad (6.6)\\ \frac{\partial c}{\partial n} &= 0 \;\text{ on } \partial\Omega\times(0,\infty). \quad (6.7)\end{aligned}$$

Here is $\Omega\subset\mathbb{R}^n$, n the exterior unit normal at the smooth parts of $\partial\Omega$, D a positive constant and $c_0(x)$ a given function.

Remark. In application to diffusion problems, $c(x,t)$ is the concentration of a substance in a solution, $c_0(x)$ its initial concentration and D the coefficient of diffusion.
First Fick's rule says that $w = D\partial c/\partial n$, where w is the flow of the substance through the boundary $\partial\Omega$. Thus according to the Neumann boundary condition (6.7), we assume that there is no flow through the boundary.

6.4.1 Fourier's method

Separation of variables ansatz $c(x,t) = v(x)w(t)$ leads to the eigenvalue problem, see the arguments of Section 4.5,

$$\begin{aligned} -\triangle v &= \lambda v \;\text{ in } \Omega \quad (6.8)\\ \frac{\partial v}{\partial n} &= 0 \;\text{ on } \partial\Omega, \quad (6.9)\end{aligned}$$

and to the ordinary differential equation

$$w'(t) + \lambda D w(t) = 0. \quad (6.10)$$

Assume Ω is bounded and $\partial\Omega$ sufficiently regular, then the eigenvalues of (6.8), (6.9) are countable and

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \ldots \to \infty.$$

Let $v_j(x)$ be a complete system of orthonormal (in $L^2(\Omega)$) eigenfunctions. Solutions of (6.10) are

$$w_j(t) = C_j \mathrm{e}^{-D\lambda_j t},$$

where C_j are arbitrary constants.

According to the superposition principle,

$$c_N(x,t) := \sum_{j=0}^{N} C_j e^{-D\lambda_j t} v_j(x)$$

is a solution of the differential equation (6.8) and

$$c(x,t) := \sum_{j=0}^{\infty} C_j e^{-D\lambda_j t} v_j(x),$$

with

$$C_j = \int_\Omega c_0(x) v_j(x)\ dx,$$

is a formal solution of the initial-boundary value problem (6.5)-(6.7).

Diffusion in a tube

Consider a solution in a tube, see Figure 6.4. Assume the initial concentration $c_0(x_1,x_2,x_3)$ of the substrate in a solution is constant if $x_3 = const$. It follows from a uniqueness result below that the solution of the initial-boundary value problem $c(x_1,x_2,x_3,t)$ is independent of x_1 and x_2.

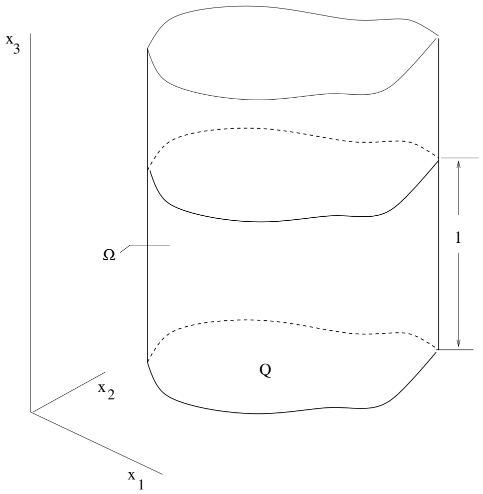

Figure 6.4: Diffusion in a tube

Set $z = x_3$, then the above initial-boundary value problem reduces to

$$\begin{aligned} c_t &= Dc_{zz} \\ c(z,0) &= c_0(z) \\ c_z &= 0, \quad z = 0,\ z = l. \end{aligned}$$

The (formal) solution is

$$c(z,t) = \sum_{n=0}^{\infty} C_n \mathrm{e}^{-D\left(\frac{\pi}{l}n\right)^2 t} \cos\left(\frac{\pi}{l}nz\right),$$

where

$$\begin{aligned} C_0 &= \frac{1}{l}\int_0^l c_0(z)\ dz \\ C_n &= \frac{2}{l}\int_0^l c_0(z)\cos\left(\frac{\pi}{l}nz\right)\ dz, \quad n \geq 1. \end{aligned}$$

6.4.2 Uniqueness

Sufficiently regular solutions of the initial-boundary value problem (6.5)-(6.7) are uniquely determined since from

$$\begin{aligned} c_t &= D\triangle c \ \text{ in } \Omega \times (0,\infty) \\ c(x,0) &= 0 \\ \frac{\partial c}{\partial n} &= 0 \ \text{ on } \partial\Omega \times (0,\infty). \end{aligned}$$

it follows that for each $\tau > 0$

$$\begin{aligned} 0 &= \int_0^\tau \int_\Omega (c_t c - D(\triangle c)c)\ dxdt \\ &= \int_\Omega \int_0^\tau \frac{1}{2}\frac{\partial}{\partial t}(c^2)\ dtdx + D\int_\Omega \int_0^\tau |\nabla_x c|^2\ dxdt \\ &= \frac{1}{2}\int_\Omega c^2(x,\tau)\ dx + D\int_\Omega \int_0^\tau |\nabla_x c|^2\ dxdt. \end{aligned}$$

6.5 Black-Scholes equation

Solutions of the Black-Scholes equation define the value of a derivative, for example of a call or put option, which is based on an asset. An asset can be a stock or a derivative again, for example. In principle, there are infinitely many such products, for example n-th derivatives. The Black-Scholes equation for the value $V(S,t)$ of a derivative is

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0 \quad \text{in } \Omega, \tag{6.11}$$

where for a fixed T, $0 < T < \infty$,

$$\Omega = \{(S,t) \in \mathbb{R}^2 : \ 0 < S < \infty, \ 0 < t < T\},$$

and σ, r are positive constants. More precisely,
σ is the volatility of the underlying asset S,
r is the guaranteed interest rate of a risk-free investment.

If $S(t)$ is the value of an asset at time t, then $V(S(t),t)$ is the value of the derivative at time t, where $V(S,t)$ is the solution of an appropriate initial-boundary value problem for the Black-Scholes equation, see below.

The Black-Scholes equation follows from Ito's Lemma under some assumptions on the random function associated to $S(t)$, see [26], for example.

Call option

Here is $V(S,t) := C(S,t)$, where $C(S,t)$ is the value of the (European) call option. In this case we have following side conditions to (6.11):

$$\begin{aligned} C(S,T) &= \max\{S-E,0\} && (6.12)\\ C(0,t) &= 0 && (6.13)\\ C(S,t) &= S + o(S) \text{ as } S \to \infty, \text{ uniformly in } t, && (6.14) \end{aligned}$$

where E and T are positive constants, E is the exercise price and T the expiry.

Side condition (6.12) means that the value of the option has no value at time T if $S(T) \le E$,
condition (6.13) says that it makes no sense to buy assets if the value of the asset is zero,
condition (6.14) means that we buy assets if its value becomes large, see Figure 6.5, where the side conditions are indicated.

Theorem 6.4 (Black-Scholes formula for European call options). *The solution $C(S,t)$, $0 \le S < \infty$, $0 \le t \le T$, of the initial-boundary value problem (6.11)-(6.14) is explicitly known and is given by*

$$C(S,t) = SN(d_1) - Ee^{-r(T-t)}N(d_2),$$

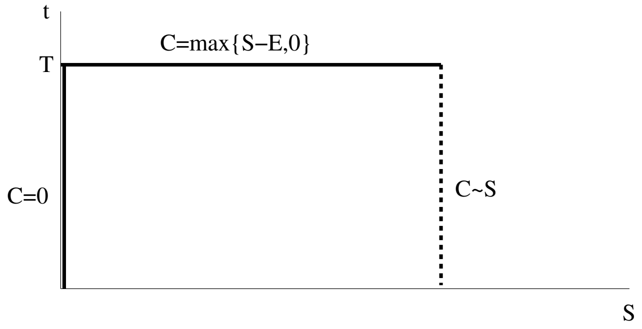

Figure 6.5: Side conditions for a call option

where

$$
\begin{aligned}
N(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy, \\
d_1 &= \frac{\ln(S/E) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \\
d_2 &= \frac{\ln(S/E) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.
\end{aligned}
$$

Proof. Substitutions

$$S = E\mathrm{e}^x, \quad t = T - \frac{\tau}{\sigma^2/2}, \quad C = Ev(x,\tau)$$

change equation (6.11) to

$$v_\tau = v_{xx} + (k-1)v_x - kv, \tag{6.15}$$

where

$$k = \frac{r}{\sigma^2/2}.$$

Initial condition (6.15) implies

$$v(x,0) = \max\{\mathrm{e}^x - 1, 0\}. \tag{6.16}$$

For a solution of (6.15) we make the ansatz

$$v = \mathrm{e}^{\alpha x + \beta\tau} u(x,\tau),$$

where α and β are constants which we will determine as follows. Inserting the ansatz into differential equation (6.15), we get

$$\beta u + u_\tau = \alpha^2 u + 2\alpha u_x + u_{xx} + (k-1)(\alpha u + u_x) - ku.$$

Set $\beta = \alpha^2 + (k-1)\alpha - k$ and choose α such that $0 = 2\alpha + (k-1)$, then $u_\tau = u_{xx}$. Thus

$$v(x,\tau) = \mathrm{e}^{-(k-1)x/2-(k+1)^2\tau/4} u(x,\tau), \tag{6.17}$$

where $u(x,\tau)$ is a solution of the initial value problem

$$\begin{aligned} u_\tau &= u_{xx}, \quad -\infty < x < \infty,\ \tau > 0 \\ u(x,0) &= u_0(x), \end{aligned}$$

with

$$u_0(x) = \max\left\{\mathrm{e}^{(k+1)x/2} - \mathrm{e}^{(k-1)x/2}, 0\right\}.$$

A solution of this initial value problem is given by Poisson's formula

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{+\infty} u_0(s)\mathrm{e}^{-(x-s)^2/(4\tau)}\, ds.$$

Changing variable by $q = (s-x)/(\sqrt{2\tau})$, we get

$$\begin{aligned} u(x,\tau) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u_0(q\sqrt{2\tau}+x)\mathrm{e}^{-q^2/2}\, dq \\ &= I_1 - I_2, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \frac{1}{\sqrt{2\pi}} \int_{-x/(\sqrt{2\tau})}^{\infty} \mathrm{e}^{(k+1)(x+q\sqrt{2\tau})}\mathrm{e}^{-q^2/2}\, dq \\ I_2 &= \frac{1}{\sqrt{2\pi}} \int_{-x/(\sqrt{2\tau})}^{\infty} \mathrm{e}^{(k-1)(x+q\sqrt{2\tau})}\mathrm{e}^{-q^2/2}\, dq. \end{aligned}$$

An elementary calculation shows that

$$\begin{aligned} I_1 &= \mathrm{e}^{(k+1)x/2+(k+1)^2\tau/4} N(d_1) \\ I_2 &= \mathrm{e}^{(k-1)x/2+(k-1)^2\tau/4} N(d_2), \end{aligned}$$

where

$$\begin{aligned} d_1 &= \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k+1)\sqrt{2\tau} \\ d_2 &= \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k-1)\sqrt{2\tau} \\ N(d_i) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_i} \mathrm{e}^{-s^2/2}\, ds, \quad i = 1,\ 2. \end{aligned}$$

Combining the formula for $u(x,\tau)$, definition (6.17) of $v(x,\tau)$ and the previous settings $x = \ln(S/E)$, $\tau = \sigma^2(T-t)/2$ and $C = Ev(x,\tau)$, we get finally the formula of of Theorem 6.4.

In general, the solution u of the initial value problem for the heat equation is not uniquely defined, see for example [10], pp. 206.

Uniqueness. The uniqueness follows from the growth assumption (6.14). Assume there are two solutions of (6.11), (6.12)-(6.14), then the difference $W(S,t)$ satisfies the differential equation (6.11) and the side conditions

$$W(S,T) = 0,\ W(0,t) = 0,\ W(S,t) = O(S) \text{ as } S \to \infty$$

uniformly in $0 \le t \le T$.

From a maximum principle consideration, see an exercise, it follows that $|W(S,t)| \le cS$ on $S \ge 0$, $0 \le t \le T$. The constant c is independent on S and t. From the definition of u we see that

$$u(x,\tau) = \frac{1}{E}\mathrm{e}^{-\alpha x - \beta\tau} W(S,t),$$

where $S = E\mathrm{e}^x$, $t = T - 2\tau/(\sigma^2)$. Thus we have the growth property

$$|u(x,\tau)| \le M\mathrm{e}^{a|x|}, \quad x \in \mathbb{R}, \tag{6.18}$$

with positive constants M and a. Then the solution of $u_\tau = u_{xx}$, in $-\infty < x < \infty$, $0 \le \tau \le \sigma^2 T/2$, with the initial condition $u(x,0) = 0$ is uniquely defined in the class of functions satisfying the growth condition (6.18), see Proposition 6.2 of this chapter. That is, $u(x,\tau) \equiv 0$. $\Box$

Put option

Here is $V(S,t) := P(S,t)$, where $P(S,t)$ is the value of the (European) put option. In this case we have following side conditions to (6.11):

$$\begin{aligned} P(S,T) &= \max\{E-S,0\} && (6.19)\\ P(0,t) &= E\mathrm{e}^{-r(T-t)} && (6.20)\\ P(S,t) &= o(S) \text{ as } S\to\infty,\ \text{uniformly in } 0\le t\le T. && (6.21) \end{aligned}$$

Here E is the exercise price and T the expiry.

Side condition (6.19) means that the value of the option has no value at time T if $S(T) \geq E$,
condition (6.20) says that it makes no sense to sell assets if the value of the asset is zero,
condition (6.21) means that it makes no sense to sell assets if its value becomes large.

Theorem 6.5 (Black-Scholes formula for European put options). *The solution $P(S,t)$, $0 < S < \infty$, $t < T$ of the initial-boundary value problem (6.11), (6.19)-(6.21) is explicitly known and is given by*

$$P(S,t) = Ee^{-r(T-t)}N(-d_2) - SN(-d_1)$$

where $N(x)$, d_1, d_2 are the same as in Theorem 6.4.

Proof. The formula for the put option follows by the same calculations as in the case of a call option or from the put-call parity

$$C(S,t) - P(S,t) = S - E\mathrm{e}^{-r(T-t)}$$

and from

$$N(x) + N(-x) = 1.$$

Concerning the put-call parity see an exercise. See also [26], pp. 40, for a heuristic argument which leads to the formula for the put-call parity. □

6.6 Exercises

1. Show that the solution $u(x,t)$ given by Poisson's formula satisfies
$$\inf_{z\in\mathbb{R}^n}\varphi(z)\le u(x,t)\le\sup_{z\in\mathbb{R}^n}\varphi(z)\ ,$$
provided $\varphi(x)$ is continuous and bounded on $\mathbb{R}^n$.

2. Solve for given $f(x)$ and $\mu\in\mathbb{R}$ the initial value problem
$$\begin{aligned} u_t+u_x+\mu u_{xxx} &= 0 \quad \text{in} \quad \mathbb{R}\times\mathbb{R}_+ \\ u(x,0) &= f(x)\ . \end{aligned}$$

3. Show by using Poisson's formula:
(i) Each function $f\in C([a,b])$ can be approximated uniformly by a sequence $f_n\in C^\infty[a,b]$.
(ii) In (i) we can choose polynomials f_n (Weierstrass's approximation theorem).

   Hint: Concerning (ii), replace the kernel $K=exp(-\frac{|y-x|^2}{4t})$ by a sequence of Taylor polynomials in the variable $z=-\frac{|y-x|^2}{4t}$.

4. Let $u(x,t)$ be a positive solution of
$$u_t=\mu u_{xx},\ t>0,$$
where μ is a constant. Show that $\theta:=-2\mu u_x/u$ is a solution of Burger's equation
$$\theta_t+\theta\theta_x=\mu\theta_{xx},\ t>0.$$

5. Assume $u_1(s,t),...,u_n(s,t)$ are solutions of $u_t=u_{ss}$. Show that $\prod_{k=1}^n u_k(x_k,t)$ is a solution of the heat equation $u_t-\triangle u=0$ in $\mathbb{R}^n\times(0,\infty)$.

6. Let A, B are real, symmetric and nonnegative matrices. Nonnegative means that all eigenvalues are nonnegative. Prove that trace $(AB)\equiv\sum_{i,j=1}^n a^{ij}b_{ij}\ge 0$.

   Hint: (i) Let $U=(z_1,\ldots,z_n)$, where z_l is an orthonormal system of eigenvectors to the eigenvalues λ_l of the matrix B. Then
$$X=U\begin{pmatrix}\sqrt{\lambda_1} & 0 & \cdots & 0\\ 0 & \sqrt{\lambda_2} & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & \sqrt{\lambda_n}\end{pmatrix}U^T$$

is a square root of B. We recall that

$$U^TBU = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \multicolumn{4}{c}{\dotfill} \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

(ii) trace (QR) =trace (RQ).
(iii) Let $\mu_1(C), \ldots \mu_n(C)$ are the eigenvalues of a real symmetric $n \times n$-matrix. Then trace $C = \sum_{l=1}^n \mu_l(C)$, which follows from the fundamental lemma of algebra:

$$\begin{aligned} \text{det } (\lambda I - C) &= \lambda^n - (c_{11} + \ldots + c_{nn})\lambda^{n-1} + \ldots \\ &\equiv (\lambda - \mu_1) \cdot \ldots \cdot (\lambda - \mu_n) \\ &= \lambda^n - (\mu_1 + \ldots + \mu_n)\lambda^{n+1} + \ldots \end{aligned}$$

7. Assume Ω is bounded, u is a solution of the heat equation and u satisfies the regularity assumptions of the maximum principle (Theorem 6.2). Show that u achieves its maximum and its minimum on S_T.

8. Prove the following comparison principle: Assume Ω is bounded and u, v satisfy the regularity assumptions of the maximum principle. Then

$$\begin{aligned} u_t - \triangle u &\le v_t - \triangle v \ \text{ in } D_T \\ u &\le v \ \text{ on } S_T \end{aligned}$$

imply that $u \le v$ in D_T.

9. Show that the comparison principle implies the maximum principle.

10. Consider the boundary-initial value problem

$$\begin{aligned} u_t - \triangle u &= f(x,t) \ \text{ in } D_T \\ u(x,t) &= \phi(x,t) \ \text{ on } S_T, \end{aligned}$$

where f, ϕ are given.
Prove uniqueness in the class $u,\ u_t,\ u_{x_ix_k} \in C(\overline{D_T})$.

11. Assume $u,\ v_1,\ v_2 \in C^2(D_T) \cap C(\overline{D_T})$, and u is a solution of the previous boundary-initial value problem and v_1, v_2 satisfy

$$\begin{aligned} (v_1)_t - \triangle v_1 &\le f(x,t) \le (v_2)_t - \triangle v_2 \ \text{ in } D_T \\ v_1 &\le \phi \le v_2 \ \text{ on } S_T. \end{aligned}$$

Show that (inclusion theorem)

$$v_1(x,t) \le u(x,t) \le v_2(x,t) \quad \text{on } \overline{D_T}.$$

12. Show by using the comparison principle: let u be a sufficiently regular solution of

$$\begin{aligned} u_t - \triangle u &= 1 \quad \text{in} \quad D_T \\ u &= 0 \quad \text{on} \quad S_T, \end{aligned}$$

then $0 \le u(x,t) \le t \quad \text{in} \quad D_T$.

13. Discuss the result of Theorem 6.3 for the case

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x,t)u_{x_i x_j} + \sum_{i}^{n} b_i(x,t)u_{x_i} + c(x,t)u(x,t).$$

14. Show that

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2 t} \sin(nx),$$

where

$$c_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\ dx,$$

is a solution of the initial-boundary value problem

$$\begin{aligned} u_t &= u_{xx},\ x \in (0,\pi),\ t > 0, \\ u(x,0) &= f(x), \\ u(0,t) &= 0, \\ u(\pi,t) &= 0, \end{aligned}$$

if $f \in C^4(\mathbb{R})$ is odd with respect to 0 and 2π-periodic.

15. (i) Find the solution of the diffusion problem $c_t = Dc_{zz}$ in $0 \le z \le l$, $0 \le t < \infty$, $D = const. > 0$, under the boundary conditions $c_z(z,t) = 0$ if $z = 0$ and $z = l$ and with the given initial concentration

$$c(z,0) = c_0(z) := \begin{cases} c_0 = const. & \text{if} \quad 0 \le z \le h \\ 0 & \text{if} \quad h < z \le l. \end{cases}$$

(ii) Calculate $\lim_{t\to\infty}\ c(z,t)$.

16. Prove the Black-Scholes Formel for an European put option.

    Hint: Put-call parity.

17. Prove the put-call parity for European options

$$C(S,t) - P(S,t) = S - E\mathrm{e}^{-r(T-t)}$$

    by using the following uniqueness result: Assume W is a solution of (6.11) under the side conditions $W(S,T) = 0$, $W(0,t) = 0$ and $W(S,t) = O(S)$ as $S \to \infty$, uniformly on $0 \leq t \leq T$. Then $W(S,t) \equiv 0$.

18. Prove that a solution $V(S,t)$ of the initial-boundary value problem (6.11) in Ω under the side conditions (i) $V(S,T) = 0$, $S \geq 0$, (ii) $V(0,t) = 0$, $0 \leq t \leq T$, (iii) $\lim_{S\to\infty} V(S,t) = 0$ uniformly in $0 \leq t \leq T$, is uniquely determined in the class $C^2(\Omega) \cap C(\overline{\Omega})$.

19. Prove that a solution $V(S,t)$ of the initial-boundary value problem (6.11) in Ω, under the side conditions (i) $V(S,T) = 0$, $S \geq 0$, (ii) $V(0,t) = 0$, $0 \leq t \leq T$, (iii) $V(S,t) = S + o(S)$ as $S \to \infty$, uniformly on $0 \leq t \leq T$, satisfies $|V(S,t)| \leq cS$ for all $S \geq 0$ and $0 \leq t \leq T$.

Chapter 7

Elliptic equations of second order

Here we consider linear elliptic equations of second order, mainly the Laplace equation

$$\triangle u = 0.$$

Solutions of the Laplace equation are called *potential functions* or *harmonic functions.* The Laplace equation is called also potential equation.

The general elliptic equation for a scalar function $u(x)$, $x \in \Omega \subset \mathbb{R}^n$, is

$$Lu := \sum_{i,j=1}^{n} a^{ij}(x) u_{x_i x_j} + \sum_{j=1}^{n} b^j(x) u_{x_j} + c(x) u = f(x),$$

where the matrix $A = (a^{ij})$ is real, symmetric and positive definite. If A is a constant matrix, then a transform to principal axis and stretching of axis leads to

$$\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} = \triangle v,$$

where $v(y) := u(Ty)$, T stands for the above composition of mappings.

7.1 Fundamental solution

Here we consider particular solutions of the Laplace equation in $\mathbb{R}^n$ of the type

$$u(x) = f(|x - y|),$$

where $y \in \mathbb{R}^n$ is fixed and f is a function which we will determine such that u defines a solution if the Laplace equation.

Set $r = |x - y|$, then

$$\begin{aligned} u_{x_i} &= f'(r)\frac{x_i - y_i}{r} \\ u_{x_i x_i} &= f''(r)\frac{(x_i - y_i)^2}{r^2} + f'(r)\left(\frac{1}{r} - \frac{(x_i - y_i)^2}{r^3}\right) \\ \triangle u &= f''(r) + \frac{n-1}{r}f'(r). \end{aligned}$$

Thus a solution of $\triangle u = 0$ is given by

$$f(r) = \left\{ \begin{array}{ccc} c_1 \ln r + c_2 & : & n = 2 \\ c_1 r^{2-n} + c_2 & : & n \geq 3 \end{array} \right.$$

with constants c_1, c_2.

Definition. Set $r = |x - y|$. The function

$$s(r) := \left\{ \begin{array}{ccc} -\frac{1}{2\pi} \ln r & : & n = 2 \\ \frac{r^{2-n}}{(n-2)\omega_n} & : & n \geq 3 \end{array} \right.$$

is called *singularity function* associated to the Laplace equation. Here is ω_n the area of the n-dimensional unit sphere which is given by $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$, where

$$\Gamma(t) := \int_0^\infty \mathrm{e}^{-\rho} \rho^{t-1}\ d\rho, \quad t > 0,$$

is the Gamma function.

Definition. A function

$$\gamma(x, y) = s(r) + \phi(x, y)$$

is called *fundamental solution* associated to the Laplace equation if $\phi \in C^2(\Omega)$ and $\triangle_x \phi = 0$ for each fixed $y \in \Omega$.

Remark. The fundamental solution γ satisfies for each fixed $y \in \Omega$ the relation

$$-\int_\Omega \gamma(x, y) \triangle_x \Phi(x)\ dx = \Phi(y) \ \text{ for all } \Phi \in C_0^2(\Omega),$$

see an exercise. This formula follows from considerations similar to the next section.

In the language of distribution, this relation can be written by definition as

$$-\triangle_x \gamma(x,y) = \delta(x-y),$$

where δ is the Dirac distribution, which is called δ-function.

7.2 Representation formula

In the following we assume that Ω, the function ϕ which appears in the definition of the fundamental solution and the potential function u considered are sufficiently regular such that the following calculations make sense, see [6] for generalizations. This is the case if Ω is bounded, $\partial\Omega$ is in C^1, $\phi \in C^2(\overline{\Omega})$ for each fixed $y \in \Omega$ and $u \in C^2(\overline{\Omega})$.

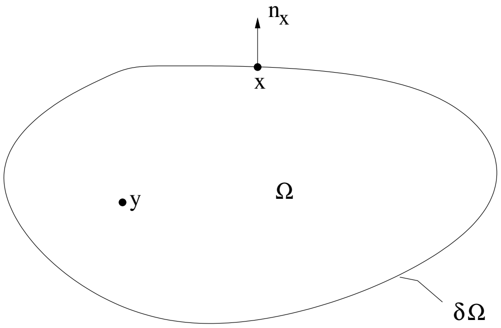

Figure 7.1: Notations to Green's identity

Theorem 7.1. *Let u be a potential function and γ a fundamental solution, then for each fixed $y \in \Omega$*

$$u(y) = \int_{\partial\Omega} \left(\gamma(x,y) \frac{\partial u(x)}{\partial n_x} - u(x) \frac{\partial \gamma(x,y)}{\partial n_x} \right) \, dS_x.$$

Proof. Let $B_\rho(y) \subset \Omega$ be a ball. Set $\Omega_\rho(y) = \Omega \setminus B_\rho(y)$. See Figure 7.2 for notations. From Green's formula, for $u,\ v \in C^2(\overline{\Omega})$,

$$\int_{\Omega_\rho(y)} (v\triangle u - u\triangle v) \, dx = \int_{\partial\Omega_\rho(y)} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n} \right) \, dS$$

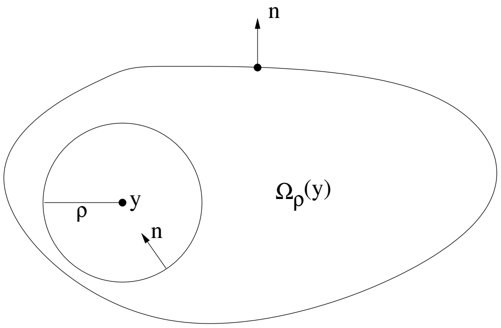

Figure 7.2: Notations to Theorem 7.1

we obtain, if v is a fundamental solution and u a potential function,

$$\int_{\partial\Omega_\rho(y)} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n} \right)\ dS = 0.$$

Thus we have to consider

$$\begin{aligned}
\int_{\partial\Omega_\rho(y)} v\frac{\partial u}{\partial n}\ dS &= \int_{\partial\Omega} v\frac{\partial u}{\partial n}\ dS + \int_{\partial B_\rho(y)} v\frac{\partial u}{\partial n}\ dS \\
\int_{\partial\Omega_\rho(y)} u\frac{\partial v}{\partial n}\ dS &= \int_{\partial\Omega} u\frac{\partial v}{\partial n}\ dS + \int_{\partial B_\rho(y)} u\frac{\partial v}{\partial n}\ dS.
\end{aligned}$$

We estimate the integrals over $\partial B_\rho(y)$:

(i)

$$\begin{aligned}
\left| \int_{\partial B_\rho(y)} v\frac{\partial u}{\partial n}\ dS \right| &\le M \int_{\partial B_\rho(y)} |v|\ dS \\
&\le M \left(\int_{\partial B_\rho(y)} s(\rho)\ dS + C\omega_n \rho^{n-1} \right),
\end{aligned}$$

where

$$\begin{aligned}
M &= M(y) = \sup_{B_{\rho_0}(y)} |\partial u/\partial n|, \quad \rho \le \rho_0, \\
C &= C(y) = \sup_{x \in B_{\rho_0}(y)} |\phi(x,y)|.
\end{aligned}$$

From the definition of $s(\rho)$ we get the estimate as $\rho\to 0$

$$\int_{\partial B_\rho(y)} v\frac{\partial u}{\partial n}\,dS=\left\{\begin{array}{rcl} O(\rho|\ln\rho|) & : & n=2\\ O(\rho) & : & n\ge 3.\end{array}\right. \tag{7.1}$$

(ii) Consider the case $n\ge 3$, then

$$\begin{aligned}\int_{\partial B_\rho(y)} u\frac{\partial v}{\partial n}\,dS &= \frac{1}{\omega_n}\int_{\partial B_\rho(y)} u\frac{1}{\rho^{n-1}}\,dS+\int_{\partial B_\rho(y)} u\frac{\partial\phi}{\partial n}\,dS\\ &= \frac{1}{\omega_n\rho^{n-1}}\int_{\partial B_\rho(y)} u\,dS+O(\rho^{n-1})\\ &= \frac{1}{\omega_n\rho^{n-1}}u(x_0)\int_{\partial B_\rho(y)} dS+O(\rho^{n-1}),\\ &= u(x_0)+O(\rho^{n-1}).\end{aligned}$$

for an $x_0\in\partial B_\rho(y)$.

Combining this estimate and (7.1), we obtain the representation formula of the theorem. □

Corollary. Set $\phi\equiv 0$ and $r=|x-y|$ in the representation formula of Theorem 7.1, then

$$u(y) = \frac{1}{2\pi}\int_{\partial\Omega}\left(\ln r\frac{\partial u}{\partial n_x}-u\frac{\partial(\ln r)}{\partial n_x}\right)\,dS_x,\quad n=2, \tag{7.2}$$

$$u(y) = \frac{1}{(n-2)\omega_n}\int_{\partial\Omega}\left(\frac{1}{r^{n-2}}\frac{\partial u}{\partial n_x}-u\frac{\partial(r^{2-n})}{\partial n_x}\right)\,dS_x,\quad n\ge 3. \tag{7.3}$$

7.2.1 Conclusions from the representation formula

Similar to the theory of functions of one complex variable, we obtain here results for harmonic functions from the representation formula, in particular from (7.2), (7.3). We recall that a function u is called *harmonic* if $u\in C^2(\Omega)$ and $\triangle u=0$ in Ω.

Proposition 7.1. *Assume u is harmonic in Ω. Then $u\in C^\infty(\Omega)$.*

Proof. Let $\Omega_0\subset\subset\Omega$ be a domain such that $y\in\Omega_0$. It follows from representation formulas (7.2), (7.3), where $\Omega:=\Omega_0$, that $D^lu(y)$ exist and are continuous for all l since one can change differentiation with integration in right hand sides of the representation formulae. $\square$

Remark. In fact, a function which is harmonic in Ω is even real analytic in Ω, see an exercise.

Proposition 7.2 (Mean value formula for harmonic functions). *Assume u is harmonic in Ω. Then for each $B_\rho(x) \subset\subset \Omega$*

$$u(x) = \frac{1}{\omega_n \rho^{n-1}} \int_{\partial B_\rho(x)} u(y)\ dS_y.$$

Proof. Consider the case $n \geq 3$. The assertion follows from (7.3) where $\Omega := B_\rho(x)$ since $r = \rho$ and

$$\begin{aligned}\int_{\partial B_\rho(x)} \frac{1}{r^{n-2}} \frac{\partial u}{\partial n_y}\ dS_y &= \frac{1}{\rho^{n-2}} \int_{\partial B_\rho(x)} \frac{\partial u}{\partial n_y}\ dS_y \\ &= \frac{1}{\rho^{n-2}} \int_{B_\rho(x)} \triangle u\ dy \\ &= 0.\end{aligned}$$

$\square$

We recall that a domain $\Omega \in \mathbb{R}^n$ is called connected if Ω is not the union of two nonempty open subsets Ω_1, Ω_2 such that $\Omega_1 \cap \Omega_2 = \emptyset$. A domain in $\mathbb{R}^n$ is connected if and only if its path connected.

Proposition 7.3 (Maximum principle). *Assume u is harmonic in a connected domain and achieves its supremum or infimum in Ω. Then $u \equiv const.$ in Ω.*

Proof. Consider the case of the supremum. Let $x_0 \in \Omega$ such that

$$u(x_0) = \sup_\Omega u(x) =: M.$$

Set $\Omega_1 := \{x \in \Omega :\ u(x) = M\}$ and $\Omega_2 := \{x \in \Omega :\ u(x) < M\}$. The set Ω_1 is not empty since $x_0 \in \Omega_1$. The set Ω_2 is open since $u \in C^2(\Omega)$. Consequently, Ω_2 is empty if we can show that Ω_1 is open. Let $\overline{x} \in \Omega_1$, then there is a $\rho_0 > 0$ such that $\overline{B_{\rho_0}(\overline{x})} \subset \Omega$ and $u(x) = M$ for all $x \in B_{\rho_0}(\overline{x})$.

If not, then there exists $\rho > 0$ and $\widehat{x}$ such that $|\widehat{x} - \overline{x}| = \rho$, $0 < \rho < \rho_0$ and $u(\widehat{x}) < M$. From the mean value formula, see Proposition 7.2, it follows

$$M = \frac{1}{\omega_n \rho^{n-1}} \int_{\partial B_\rho(\overline{x})} u(x)\ dS < \frac{M}{\omega_n \rho^{n-1}} \int_{\partial B_\rho(\overline{x})} dS = M,$$

which is a contradiction. Thus, the set Ω_2 is empty since Ω_1 is open. □

Corollary. Assume Ω is connected and bounded, and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is harmonic in Ω. Then u achieves its minimum and its maximum on the boundary $\partial\Omega$.

Remark. The previous corollary fails if Ω is not bounded as simple counterexamples show.

7.3 Boundary value problems

Assume $\Omega \subset \mathbb{R}^n$ is a connected domain.

7.3.1 Dirichlet problem

The *Dirichlet problem* (first boundary value problem) is to find a solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of

$$\begin{aligned} \triangle u &= 0 \ \text{ in } \Omega && (7.4)\\ u &= \Phi \ \text{ on } \partial\Omega, && (7.5) \end{aligned}$$

where Φ is given and continuous on $\partial\Omega$.

Proposition 7.4. *Assume Ω is bounded, then a solution to the Dirichlet problem is uniquely determined.*

Proof. Maximum principle.

Remark. The previous result fails if we take away in the boundary condition (7.5) one point from the the boundary as the following example shows. Let $\Omega \subset \mathbb{R}^2$ be the domain

$$\Omega = \{x \in B_1(0) :\ x_2 > 0\},$$

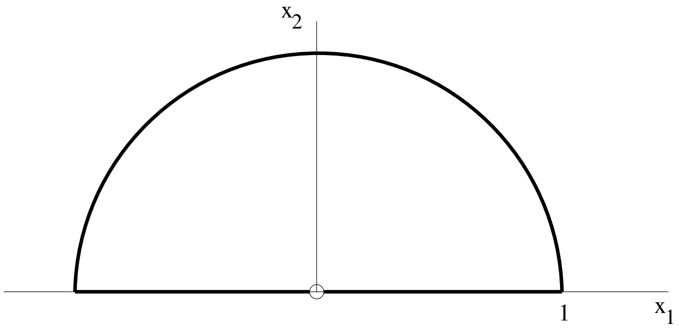

Figure 7.3: Counterexample

Assume $u \in C^2(\Omega) \cap C(\overline{\Omega} \setminus \{0\})$ is a solution of

$$\begin{aligned} \triangle u &= 0 \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \setminus \{0\}. \end{aligned}$$

This problem has solutions $u \equiv 0$ *and* $u = \mathrm{Im}(z + z^{-1})$, where $z = x_1 + ix_2$. Another example see an exercise.

In contrast to this behaviour of the Laplace equation, one has uniqueness if $\triangle u = 0$ is replaced by the minimal surface equation

$$\frac{\partial}{\partial x_1}\left(\frac{u_{x_1}}{\sqrt{1+|\nabla u|^2}}\right) + \frac{\partial}{\partial x_2}\left(\frac{u_{x_2}}{\sqrt{1+|\nabla u|^2}}\right) = 0.$$

7.3.2 Neumann problem

The *Neumann problem* (second boundary value problem) is to find a solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ of

$$\begin{aligned} \triangle u &= 0 \text{ in } \Omega && (7.6) \\ \frac{\partial u}{\partial n} &= \Phi \text{ on } \partial\Omega, && (7.7) \end{aligned}$$

where Φ is given and continuous on $\partial\Omega$.

Proposition 7.5. *Assume Ω is bounded, then a solution to the Dirichlet problem is in the class $u \in C^2(\overline{\Omega})$ uniquely determined up to a constant.*

Proof. Exercise. Hint: Multiply the differential equation $\triangle w = 0$ by w and integrate the result over Ω.

Another proof under the weaker assumption $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ follows from the Hopf boundary point lemma, see Lecture Notes: Linear Elliptic Equations of Second Order, for example.

7.3.3 Mixed boundary value problem

The *Mixed boundary value problem* (third boundary value problem) is to find a solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ of

$$\begin{aligned} \triangle u &= 0 \ \text{ in } \Omega && (7.8)\\ \frac{\partial u}{\partial n} + hu &= \Phi \ \text{ on } \partial\Omega, && (7.9) \end{aligned}$$

where Φ and h are given and continuous on $\partial\Omega$.e Φ and h are given and continuous on $\partial\Omega$.

Proposition 7.6. *Assume Ω is bounded and sufficiently regular, then a solution to the mixed problem is uniquely determined in the class $u \in C^2(\overline{\Omega})$ provided $h(x) \geq 0$ on $\partial\Omega$ and $h(x) > 0$ for at least one point $x \in \partial\Omega$.*

Proof. Exercise. Hint: Multiply the differential equation $\triangle w = 0$ by w and integrate the result over Ω.

7.4 Green's function for △

Theorem 7.1 says that each harmonic function satisfies

$$u(x) = \int_{\partial\Omega} \left(\gamma(y,x) \frac{\partial u(y)}{\partial n_y} - u(y) \frac{\partial \gamma(y,x)}{\partial n_y} \right) \, dS_y, \tag{7.10}$$

where $\gamma(y,x)$ is a fundamental solution. In general, u does not satisfies the boundary condition in the above boundary value problems. Since $\gamma = s + \phi$, see Section 7.2, where ϕ is an *arbitrary* harmonic function for each fixed x, we try to find a ϕ such that u satisfies also the boundary condition.

Consider the Dirichlet problem, then we look for a ϕ such that

$$\gamma(y,x) = 0, \quad y \in \partial\Omega, \ x \in \Omega. \tag{7.11}$$

Then

$$u(x) = -\int_{\partial\Omega} \frac{\partial \gamma(y,x)}{\partial n_y} u(y) \, dS_y, \quad x \in \Omega.$$

Suppose that u achieves its boundary values Φ of the Dirichlet problem, then

$$u(x) = -\int_{\partial\Omega} \frac{\partial\gamma(y,x)}{\partial n_y}\Phi(y)\ dS_y, \tag{7.12}$$

We claim that this function solves the Dirichlet problem (7.4), (7.5).

A function $\gamma(y,x)$ which satisfies (7.11), and some additional assumptions, is called *Green's* function. More precisely, we define a Green function as follows.

Definition. A function $G(y,x)$, $y,\ x \in \overline{\Omega}$, $x \neq y$, is called *Green function* associated to Ω and to the Dirichlet problem (7.4), (7.5) if for fixed $x \in \Omega$, that is we consider $G(y,x)$ as a function of y, the following properties hold:

(i) $G(y,x) \in C^2(\Omega \setminus \{x\}) \cap C(\overline{\Omega} \setminus \{x\}), \qquad \triangle_y G(y,x) = 0, \quad x \neq y.$

(ii) $G(y,x) - s(|x-y|) \in C^2(\Omega) \cap C(\overline{\Omega})$.

(iii) $G(y,x) = 0$ if $y \in \partial\Omega$, $x \neq y$.

Remark. We will see in the next section that a Green function exists at least for some domains of simple geometry. Concerning the existence of a Green function for more general domains see [13].

It is an interesting fact that we get from (i)-(iii) of the above definition two further important properties. We assume that Ω is bounded, sufficiently regular and connected.

Proposition 7.7. *A Green function has the following properties. In the case $n = 2$ we assume* diam $\Omega < 1$.

(A) $G(x,y) = G(y,x)$ *(symmetry).*

(B) $0 < G(x,y) < s(|x-y|), \quad x,\ y \in \Omega,\ x \neq y.$

Proof. (A) Let $x^{(1)},\ x^{(2)} \in \Omega$. Set $B_i = B_\rho(x^{(i)})$, $i = 1,\ 2$. We assume $\overline{B_i} \subset \Omega$ and $B_1 \cap B_2 = \emptyset$. Since $G(y,x^{(1)})$ and $G(y,x^{(2)})$ are harmonic in $\Omega \setminus \left(\overline{B_1} \cup \overline{B_2}\right)$ we obtain from Green's identity, see Figure 7.4 for notations,

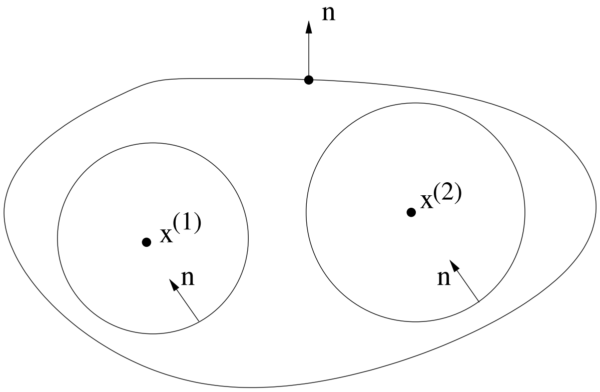

Figure 7.4: Proof of Proposition 7.7

$$\begin{aligned}
0 &= \int_{\partial\left(\Omega\setminus(\overline{B_1}\cup\overline{B_2})\right)}\Big(G(y,x^{(1)})\frac{\partial}{\partial n_y}G(y,x^{(2)})\\
&\qquad\qquad -G(y,x^{(2)})\frac{\partial}{\partial n_y}G(y,x^{(1)})\Big)dS_y\\
&= \int_{\partial\Omega}\left(G(y,x^{(1)})\frac{\partial}{\partial n_y}G(y,x^{(2)})-G(y,x^{(2)})\frac{\partial}{\partial n_y}G(y,x^{(1)})\right)dS_y\\
&+ \int_{\partial B_1}\left(G(y,x^{(1)})\frac{\partial}{\partial n_y}G(y,x^{(2)})-G(y,x^{(2)})\frac{\partial}{\partial n_y}G(y,x^{(1)})\right)dS_y\\
&+ \int_{\partial B_2}\left(G(y,x^{(1)})\frac{\partial}{\partial n_y}G(y,x^{(2)})-G(y,x^{(2)})\frac{\partial}{\partial n_y}G(y,x^{(1)})\right)dS_y.
\end{aligned}$$

The integral over $\partial\Omega$ is zero because of property (iii) of a Green function, and

$$\begin{aligned}
\int_{\partial B_1}\Big(G(y,x^{(1)})\frac{\partial}{\partial n_y}G(y,x^{(2)}) &- G(y,x^{(2)})\frac{\partial}{\partial n_y}G(y,x^{(1)})\Big)dS_y\\
&\to G(x^{(1)},x^{(2)}),\\
\int_{\partial B_2}\Big(G(y,x^{(1)})\frac{\partial}{\partial n_y}G(y,x^{(2)}) &- G(y,x^{(2)})\frac{\partial}{\partial n_y}G(y,x^{(1)})\Big)dS_y\\
&\to -G(x^{(2)},x^{(1)})
\end{aligned}$$

as $\rho\to 0$. This follows by considerations as in the proof of Theorem 7.1.

(B) Since

$$G(y,x)=s(|x-y|)+\phi(y,x)$$

and $G(y,x)=0$ if $y\in\partial\Omega$ and $x\in\Omega$ we have for $y\in\partial\Omega$

$$\phi(y,x)=-s(|x-y|).$$

From the definition of $s(|x-y|)$ it follows that $\phi(y,x)<0$ if $y\in\partial\Omega$. Thus, since $\triangle_y\phi=0$ in Ω, the maximum-minimum principle implies that $\phi(y,x)<0$ for all $y,\ x\in\Omega$. Consequently

$$G(y,x)<s(|x-y|),\quad x,\ y\in\Omega,\ x\neq y.$$

It remains to show that

$$G(y,x)>0,\quad x,\ y\in\Omega,\ x\neq y.$$

Fix $x\in\Omega$ and let $B_\rho(x)$ be a ball such that $B_\rho(x)\subset\Omega$ for all $0<\rho<\rho_0$. There is a sufficiently small $\rho_0>0$ such that for each ρ, $0<\rho<\rho_0$,

$$G(y,x)>0\ \text{ for all } y\in\overline{B_\rho(x)},\ x\neq y,$$

see property (iii) of a Green function. Since

$$\begin{array}{rcl}\triangle_y G(y,x) &=& 0\ \text{ in } \Omega\setminus\overline{B_\rho(x)}\\ G(y,x) &>& 0\ \text{ if } y\in\partial B_\rho(x)\\ G(y,x) &=& 0\ \text{ if } y\in\partial\Omega\end{array}$$

it follows from the maximum-minimum principle that

$$G(y,x)>0\ \text{ on } \Omega\setminus\overline{B_\rho(x)}.$$

□

7.4.1 Green's function for a ball

If $\Omega=B_R(0)$ is a ball, then Green's function is explicitly known.

Let $\Omega=B_R(0)$ be a ball in $\mathbb{R}^n$ with radius R and the center at the origin. Let $x,\ y\in B_R(0)$ and let y' the reflected point of y on the sphere $\partial B_R(0)$, that is, in particular $|y||y'|=R^2$, see Figure 7.5 for notations. Set

$$G(x,y)=s(r)-s\left(\frac{\rho}{R}r_1\right),$$

where s is the singularity function of Section 7.1, $r=|x-y|$ and

$$\rho^2=\sum_{i=1}^n y_i^2,\quad r_1=\sum_{i=1}^n\left(x_i-\frac{R^2}{\rho^2}y_i\right)^2.$$

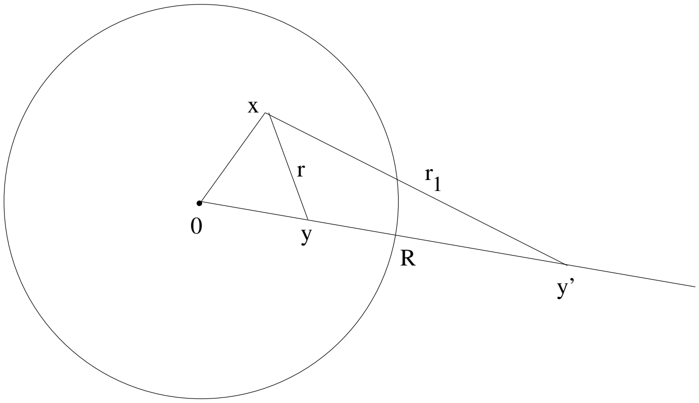

Figure 7.5: Reflection on $\partial B_R(0)$

This function $G(x,y)$ satisfies (i)-(iii) of the definition of a Green function. We claim that

$$u(x) = -\int_{\partial B_R(0)} \frac{\partial}{\partial n_y} G(x,y)\Phi\ dS_y$$

is a solution of the Dirichlet problem (7.4), (7.5). This formula is also true for a large class of domains $\Omega \subset \mathbb{R}^n$, see [13].

Lemma.

$$-\frac{\partial}{\partial n_y} G(x,y)\bigg|_{|y|=R} = \frac{1}{R\omega_n}\frac{R^2-|x|^2}{|y-x|^n}.$$

Proof. Exercise.

Set

$$H(x,y) = \frac{1}{R\omega_n}\frac{R^2-|x|^2}{|y-x|^n}, \tag{7.13}$$

which is called *Poisson's kernel.*

Theorem 7.2. *Assume* $\Phi \in C(\partial\Omega)$. *Then*

$$u(x) = \int_{\partial B_R(0)} H(x,y)\Phi(y)\ dS_y$$

is the solution of the first boundary value problem (7.4), (7.5) in the class $C^2(\Omega)\cap C(\overline{\Omega})$.

Proof. The proof follows from following properties of $H(x,y)$:

(i) $H(x,y) \in C^\infty, \quad |y| = R,\ |x| < R,\ x \neq y,$

(ii) $\triangle_x H(x,y) = 0, \quad |x| < R,\ |y| = R,$

(iii) $\int_{\partial B_R(0)} H(x,y)\ dS_y = 1, \quad |x| < R,$

(iv) $H(x,y) > 0, \quad |y| = R,\ |x| < R,$

(v) Fix $\zeta \in \partial B_R(0)$ and $\delta > 0$, then $\lim_{x \to \zeta, |x|<R} H(x,y) = 0$ uniformly in $y \in \partial B_R(0),\ |y - \zeta| > \delta.$

(i), (iv) and (v) follow from the definition (7.13) of H and (ii) from (7.13) or from

$$H = -\left.\frac{\partial G(x,y)}{\partial n_y}\right|_{y \in \partial B_R(0)},$$

G harmonic and $G(x,y) = G(y,x)$.
Property (iii) is a consequence of formula

$$u(x) = \int_{\partial B_R(0)} H(x,y) u(y)\ dS_y,$$

for each harmonic function u, see calculations to the representation formula above. We obtain (ii) if we set $u \equiv 1$.

It remains to show that u, given by Poisson's formula, is in $C(\overline{B_R(0)})$ and that u achieves the prescribed boundary values. Fix $\zeta \in \partial B_R(0)$ and let $x \in B_R(0)$. Then

$$\begin{aligned} u(x) - \Phi(\zeta) &= \int_{\partial B_R(0)} H(x,y) \left(\Phi(y) - \Phi(\zeta)\right)\ dS_y \\ &= I_1 + I_2, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_{\partial B_R(0),\ |y-\zeta|<\delta} H(x,y) \left(\Phi(y) - \Phi(\zeta)\right)\ dS_y \\ I_2 &= \int_{\partial B_R(0),\ |y-\zeta|\geq\delta} H(x,y) \left(\Phi(y) - \Phi(\zeta)\right)\ dS_y. \end{aligned}$$

For given (small) $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ such that

$$|\Phi(y) - \Phi(\zeta)| < \epsilon$$

for all $y \in \partial B_R(0)$ with $|y - \zeta| < \delta$. It follows $|I_1| \leq \epsilon$ because of (iii) and (iv).
Set $M = \max_{\partial B_R(0)} |\phi|$. From (v) we conclude that there is a $\delta' > 0$ such that

$$H(x,y) < \frac{\epsilon}{2M\omega_n R^{n-1}}$$

if x and y satisfy $|x-\zeta| < \delta',\ |y-\zeta| > \delta$, see Figure 7.6 for notations. Thus

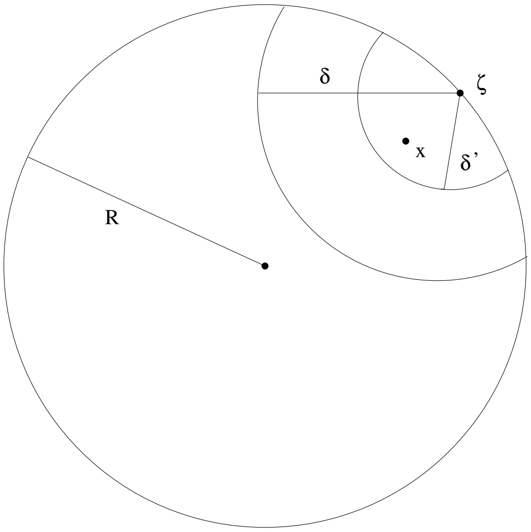

Figure 7.6: Proof of Theorem 7.2

$|I_2| < \epsilon$ and the inequality

$$|u(x) - \Phi(\zeta)| < 2\epsilon$$

for $x \in B_R(0)$ such that $|x - \zeta| < \delta'$ is shown. $\square$

Remark. Define $\delta \in [0, \pi]$ through $\cos\delta = x \cdot y/(|x||y|)$, then we write Poisson's formula of Theorem 7.2 as

$$u(x) = \frac{R^2 - |x|^2}{\omega_n R} \int_{\partial B_R(0)} \Phi(y) \frac{1}{(|x|^2 + R^2 - 2|x|R\cos\delta)^{n/2}}\ dS_y.$$

In the case $n = 2$ we can expand this integral in a power series with respect to $\rho := |x|/R$ if $|x| < R$, since

$$\begin{aligned}\frac{R^2 - |x|^2}{|x| + R^2 - 2|x|R\cos\delta} &= \frac{1-\rho^2}{\rho^2 - 2\rho\cos\delta + 1}\\ &= 1 + 2\sum_{n=1}^{\infty}\rho^n\cos(n\delta),\end{aligned}$$

see [16], pp. 18 for an easy proof of this formula, or [4], Vol. II, p. 246.

7.4.2 Green's function and conformal mapping

For two-dimensional domains there is a beautiful connection between conformal mapping and Green's function. Let $w = f(z)$ be a conformal mapping from a sufficiently regular connected domain in $\mathbb{R}^2$ onto the interior of the unit circle, see Figure 7.7. Then the Green function of Ω is, see for example [16] or other text books about the theory of functions of one complex variable,

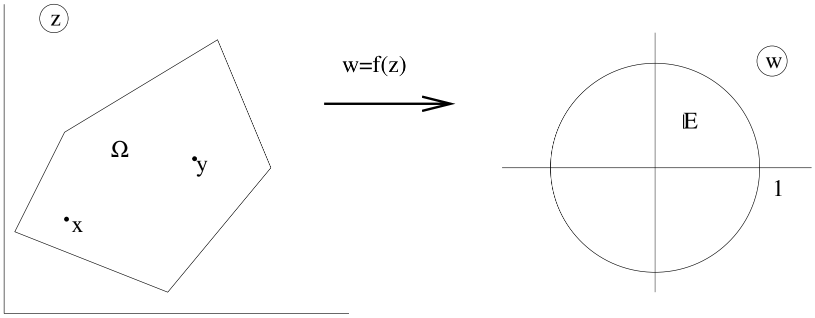

Figure 7.7: Conformal mapping

$$G(z, z_0) = \frac{1}{2\pi}\ln\left|\frac{1 - f(z)\overline{f(z_0)}}{f(z) - f(z_0)}\right|,$$

where $z = x_1 + ix_2$, $z_0 = y_1 + iy_2$.

7.5 Inhomogeneous equation

Here we consider solutions $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of

$$\begin{aligned}-\triangle u &= f(x) \text{ in } \Omega && (7.14)\\ u &= 0 \quad \text{on } \partial\Omega, && (7.15)\end{aligned}$$

where f is given.

We need the following lemma concerning volume potentials. We assume that Ω is bounded and sufficiently regular such that all the following integrals exist. See [6] for generalizations concerning these assumptions.

Let for $x \in \mathbb{R}^n$, $n \geq 3$,

$$V(x) = \int_\Omega f(y) \frac{1}{|x-y|^{n-2}}\ dy$$

and set in the two-dimensional case

$$V(x) = \int_\Omega f(y) \ln\left(\frac{1}{|x-y|}\right)\ dy.$$

We recall that $\omega_n = |\partial B_1(0)|$.

Lemma.
(i) *Assume* $f \in C(\Omega)$. *Then* $V \in C^1(\mathbb{R}^n)$ *and*

$$\begin{aligned} V_{x_i}(x) &= \int_\Omega f(y) \frac{\partial}{\partial x_i}\left(\frac{1}{|x-y|^{n-2}}\right)\ dy, \quad \textit{if } n \geq 3, \\ V_{x_i}(x) &= \int_\Omega f(y) \frac{\partial}{\partial x_i}\left(\ln\left(\frac{1}{|x-y|}\right)\right)\ dy \quad \textit{if } n = 2. \end{aligned}$$

(ii) *If* $f \in C^1(\Omega)$, *then* $V \in C^2(\Omega)$ *and*

$$\begin{aligned} \triangle V &= -(n-2)\omega_n f(x),\ x \in \Omega,\ n \geq 3 \\ \triangle V &= -2\pi f(x),\ x \in \Omega,\ n = 2. \end{aligned}$$

Proof. To simplify the presentation, we consider the case $n = 3$.
(i) The first assertion follows since we can change differentiation with integration since the differentiate integrand is weakly singular, see an exercise.
(ii) We will differentiate at $x \in \Omega$. Let B_ρ be a fixed ball such that $x \in B_\rho$, ρ sufficiently small such that $B_\rho \subset \Omega$. Then, according to (i) and since we have the identity

$$\frac{\partial}{\partial x_i}\left(\frac{1}{|x-y|}\right) = -\frac{\partial}{\partial y_i}\left(\frac{1}{|x-y|}\right)$$

which implies that

$$f(y)\frac{\partial}{\partial x_i}\left(\frac{1}{|x-y|}\right) = -\frac{\partial}{\partial y_i}\left(f(y)\frac{1}{|x-y|}\right) + f_{y_i}(y)\frac{1}{|x-y|},$$

we obtain

$$\begin{aligned}
V_{x_i}(x) &= \int_\Omega f(y)\frac{\partial}{\partial x_i}\left(\frac{1}{|x-y|}\right)\ dy\\
&= \int_{\Omega\setminus B_\rho} f(y)\frac{\partial}{\partial x_i}\left(\frac{1}{|x-y|}\right)\ dy + \int_{B_\rho} f(y)\frac{\partial}{\partial x_i}\left(\frac{1}{|x-y|}\right)\ dy\\
&= \int_{\Omega\setminus B_\rho} f(y)\frac{\partial}{\partial x_i}\left(\frac{1}{|x-y|}\right)\ dy\\
&\quad +\int_{B_\rho}\left(-\frac{\partial}{\partial y_i}\left(f(y)\frac{1}{|x-y|}\right)+f_{y_i}(y)\frac{1}{|x-y|}\right)\ dy\\
&= \int_{\Omega\setminus B_\rho} f(y)\frac{\partial}{\partial x_i}\left(\frac{1}{|x-y|}\right)\ dy\\
&\quad +\int_{B_\rho} f_{y_i}(y)\frac{1}{|x-y|}\ dy - \int_{\partial B_\rho} f(y)\frac{1}{|x-y|}n_i\ dS_y,
\end{aligned}$$

where n is the exterior unit normal at ∂B_ρ. It follows that the first and second integral is in $C^1(\Omega)$. The second integral is also in $C^1(\Omega)$ according to (i) and since $f \in C^1(\Omega)$ by assumption.

Because of $\triangle_x(|x-y|^{-1}) = 0,\ x \neq y$, it follows

$$\begin{aligned}
\triangle V &= \int_{B_\rho}\sum_{i=1}^n f_{y_i}(y)\frac{\partial}{\partial x_i}\left(\frac{1}{|x-y|}\right)\ dy\\
&\quad -\int_{\partial B_\rho} f(y)\sum_{i=1}^n\frac{\partial}{\partial x_i}\left(\frac{1}{|x-y|}\right)n_i\ dS_y.
\end{aligned}$$

Now we choose for B_ρ a ball with the center at x, then

$$\triangle V = I_1 + I_2,$$

where

$$\begin{aligned}
I_1 &= \int_{B_\rho(x)}\sum_{i=1}^n f_{y_i}(y)\frac{y_i-x_i}{|x-y|^3}\ dy\\
I_2 &= -\int_{\partial B_\rho(x)} f(y)\frac{1}{\rho^2}\ dS_y.
\end{aligned}$$

We recall that $n\cdot(y-x)=\rho$ if $y\in\partial B_\rho(x)$. It is $I_1 = O(\rho)$ as $\rho\to 0$ and for I_2 we obtain from the mean value theorem of the integral calculus that for a $\overline{y} \in \partial B_\rho(x)$

$$\begin{aligned} I_2 &= -\frac{1}{\rho^2} f(\overline{y}) \int_{\partial B_\rho(x)} dS_y \\ &= -\omega_n f(\overline{y}), \end{aligned}$$

which implies that $\lim_{\rho\to 0} I_2 = -\omega_n f(x)$. □

In the following we assume that Green's function exists for the domain Ω, which is the case if Ω is a ball.

Theorem 7.3. *Assume $f \in C^1(\Omega) \cap C(\overline{\Omega})$. Then*

$$u(x) = \int_\Omega G(x,y) f(y)\ dy$$

is the solution of the inhomogeneous problem (7.14), (7.15).

Proof. For simplicity of the presentation let $n = 3$. We will show that

$$u(x) := \int_\Omega G(x,y) f(y)\ dy$$

is a solution of (7.4), (7.5). Since

$$G(x,y) = \frac{1}{4\pi|x-y|} + \phi(x,y),$$

where ϕ is a potential function with respect to x or y, we obtain from the above lemma that

$$\begin{aligned} \triangle u &= \frac{1}{4\pi} \triangle \int_\Omega f(y) \frac{1}{|x-y|}\ dy + \int_\Omega \triangle_x \phi(x,y) f(y)\ dy \\ &= -f(x), \end{aligned}$$

where $x \in \Omega$. It remains to show that u achieves its boundary values. That is, for fixed $x_0 \in \partial\Omega$ we will prove that

$$\lim_{x\to x_0,\ x\in\Omega} u(x) = 0.$$

Set

$$u(x) = I_1 + I_2,$$

where

$$\begin{aligned} I_1(x) &= \int_{\Omega\setminus B_\rho(x_0)} G(x,y)f(y)\ dy,\\ I_2(x) &= \int_{\Omega\cap B_\rho(x_0)} G(x,y)f(y)\ dy. \end{aligned}$$

Let $M=\max_{\overline{\Omega}}|f(x)|$. Since

$$G(x,y)=\frac{1}{4\pi}\frac{1}{|x-y|}+\phi(x,y),$$

we obtain, if $x\in B_\rho(x_0)\cap\Omega$,

$$\begin{aligned} |I_2| &\le \frac{M}{4\pi}\int_{\Omega\cap B_\rho(x_0)}\frac{dy}{|x-y|}+O(\rho^2)\\ &\le \frac{M}{4\pi}\int_{B_{2\rho(x)}}\frac{dy}{|x-y|}+O(\rho^2)\\ &= O(\rho^2) \end{aligned}$$

as $\rho\to 0$. Consequently for given ϵ there is a $\rho_0=\rho_0(\epsilon)>0$ such that

$$|I_2|<\frac{\epsilon}{2}\ \text{ for all }\ 0<\rho\le\rho_0.$$

For each fixed ρ, $0<\rho\le\rho_0$, we have

$$\lim_{x\to x_0,\ x\in\Omega} I_1(x)=0$$

since $G(x_0,y)=0$ if $y\in\Omega\setminus B_\rho(x_0)$ and $G(x,y)$ is uniformly continuous in $x\in B_{\rho/2}(x_0)\cap\Omega$ and $y\in\Omega\setminus B_\rho(x_0)$, see Figure 7.8. □

Remark. For the proof of (ii) in the above lemma it is sufficient to assume that f is Hölder continuous. More precisely, let $f\in C^\lambda(\Omega)$, $0<\lambda<1$, then $V\in C^{2,\lambda}(\Omega)$, see for example [9].

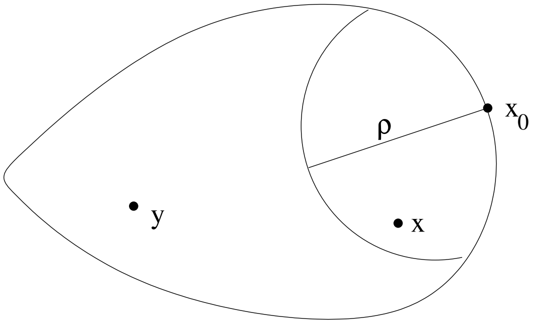

Figure 7.8: Proof of Theorem 7.3

7.6 Exercises

1. Let $\gamma(x,y)$ be a fundamental solution to $\triangle$, $y \in \Omega$. Show that

$$-\int_\Omega \gamma(x,y)\ \triangle\Phi(x)\ dx = \Phi(y) \quad \text{for all} \quad \Phi \in C_0^2(\Omega)\ .$$

Hint: See the proof of the representation formula.

2. Show that $|x|^{-1}\sin(k|x|)$ is a solution of the Helmholtz equation

$$\triangle u + k^2 u = 0 \text{ in } \mathbb{R}^n \setminus \{0\}.$$

3. Assume $u \in C^2(\overline{\Omega})$, Ω bounded and sufficiently regular, is a solution of

$$\begin{aligned} \triangle u &= u^3 \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Show that $u = 0$ in Ω.

4. Let $\Omega_\alpha = \{x \in \mathbb{R}^2 :\ x_1 > 0,\ 0 < x_2 < x_1 \tan\alpha\},\ 0 < \alpha \le \pi$. Show that

$$u(x) = r^{\frac{\pi}{\alpha}k} \sin\left(\frac{\pi}{\alpha}k\theta\right)$$

is a harmonic function in Ω_α satisfying $u = 0$ on $\partial\Omega_\alpha$, provided k is an integer. Here (r,θ) are polar coordinates with the center at $(0,0)$.

5. Let $u \in C^2(\overline{\Omega})$ be a solution of $\triangle u = 0$ on the quadrangle $\Omega = (0,1) \times (0,1)$ satisfying the boundary conditions $u(0,y) = u(1,y) = 0$ for all $y \in [0,1]$ and $u_y(x,0) = u_y(x,1) = 0$ for all $x \in [0,1]$. Prove that $u \equiv 0$ in $\overline{\Omega}$.

6. Let $u \in C^2(\mathbb{R}^n)$ be a solution of $\triangle u = 0$ in $\mathbb{R}^n$ satisfying $u \in L^2(\mathbb{R}^n)$, i. e., $\int_{\mathbb{R}^n} u^2(x)\, dx < \infty$.
   Show that $u \equiv 0$ in $\mathbb{R}^n$.

   Hint: Prove
   $$\int_{B_R(0)} |\nabla u|^2\, dx \le \frac{const.}{R^2} \int_{B_{2R}(0)} |u|^2\, dx,$$
   where c is a constant independent of R.
   To show this inequality, multiply the differential equation by $\zeta := \eta^2 u$, where $\eta \in C^1$ is a cut-off function with properties: $\eta \equiv 1$ in $B_R(0)$, $\eta \equiv 0$ in the exterior of $B_{2R}(0)$, $0 \le \eta \le 1$, $|\nabla \eta| \le C/R$. Integrate the product, apply integration by parts and use the formula $2ab \le \epsilon a^2 + \frac{1}{\epsilon} b^2$, $\epsilon > 0$.

7. Show that a bounded harmonic function defined on $\mathbb{R}^n$ must be a constant (a theorem of Liouville).

8. Assume $u \in C^2(B_1(0)) \cap C(\overline{B_1(0)} \setminus \{(1,0)\})$ is a solution of
   $$\begin{aligned} \triangle u &= 0 \text{ in } B_1(0) \\ u &= 0 \text{ on } \partial B_1(0) \setminus \{(1,0)\}. \end{aligned}$$
   Show that there are at least two solutions.

   Hint: Consider
   $$u(x,y) = \frac{1 - (x^2 + y^2)}{(1-x)^2 + y^2}.$$

9. Assume $\Omega \subset \mathbb{R}^n$ is bounded and $u,\ v \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy $\triangle u = \triangle v$ and $\max_{\partial\Omega} |u - v| \le \epsilon$ for given $\epsilon > 0$. Show that $\max_{\overline{\Omega}} |u - v| \le \epsilon$.

10. Set $\Omega = \mathbb{R}^n \setminus \overline{B_1(0)}$ and let $u \in C^2(\overline{\Omega})$ be a harmonic function in Ω satisfying $\lim_{|x| \to \infty} u(x) = 0$. Prove that
    $$\max_{\overline{\Omega}} |u| = \max_{\partial\Omega} |u| \ .$$

    Hint: Apply the maximum principle to $\Omega \cap B_R(0)$, R large.

11. Let $\Omega_\alpha = \{x \in \mathbb{R}^2 : \ x_1 > 0, \ 0 < x_2 < x_1 \tan\alpha\}, \ 0 < \alpha \leq \pi$, $\Omega_{\alpha,R} = \Omega_\alpha \cap B_R(0)$, and assume f is given and bounded on $\overline{\Omega_{\alpha,R}}$.

    Show that for each solution $u \in C^1(\overline{\Omega_{\alpha,R}}) \cap C^2(\Omega_{\alpha,R})$ of $\triangle u = f$ in $\Omega_{\alpha,R}$ satisfying $u = 0$ on $\partial\Omega_{\alpha,R} \cap B_R(0)$, holds:

    For given $\epsilon > 0$ there is a constant $C(\epsilon)$ such that

    $$|u(x)| \leq C(\epsilon)\ |x|^{\frac{\pi}{\alpha}-\epsilon} \qquad \text{in } \Omega_{\alpha,R}.$$

    Hint: (a) Comparison principle (a consequence from the maximum principle): Assume Ω is bounded, $u, v \in C^2(\overline{\Omega}) \cap C(\overline{\Omega})$ satisfying $-\triangle u \leq -\triangle v$ in Ω and $u \leq v$ on $\partial\Omega$. Then $u \leq v$ in Ω.

    (b) An appropriate comparison function is

    $$v = Ar^{\frac{\pi}{\alpha}-\epsilon}\sin(B(\theta+\eta))\ ,$$

    $A,\ B,\ \eta$ appropriate constants, $B,\ \eta$ positive.

12. Let Ω be the quadrangle $(-1,1) \times (-1,1)$ and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ a solution of the boundary value problem $-\triangle u = 1$ in Ω, $u = 0$ on $\partial\Omega$. Find a lower and an upper bound for $u(0,0)$.

    Hint: Consider the comparison function $v = A(x^2 + y^2)$, $A = const$.

13. Let $u \in C^2(B_a(0)) \cap C(\overline{B_a(0)})$ satisfying $u \geq 0,\ \triangle u = 0$ in $B_a(0)$. Prove (Harnack's inequality):

    $$\frac{a^{n-2}(a-|\zeta|)}{(a+|\zeta|)^{n-1}}u(0) \leq u(\zeta) \leq \frac{a^{n-2}(a+|\zeta|)}{(a-|\zeta|)^{n-1}}u(0)\ .$$

    Hint: Use the formula (see Theorem 7.2)

    $$u(y) = \frac{a^2-|y|^2}{a\omega_n}\int_{|x|=a}\frac{u(x)}{|x-y|^n}\,dS_x$$

    for $y = \zeta$ and $y = 0$.

14. Let $\phi(\theta)$ be a 2π-periodic C^4-function with the Fourier series

    $$\phi(\theta) = \sum_{n=0}^{\infty}\left(a_n\ cos(n\theta) + b_n\sin(n\theta)\right)\ .$$

    Show that

    $$u = \sum_{n=0}^{\infty}\left(a_n\ cos(n\theta) + b_n\sin(n\theta)\right)r^n$$

    solves the Dirichlet problem in $B_1(0)$.

15. Assume $u \in C^2(\Omega)$ satisfies $\triangle u = 0$ in Ω. Let $B_a(\zeta)$ be a ball such that its closure is in Ω. Show that

$$|D^\alpha u(\zeta)| \le M \left(\frac{|\alpha|\gamma_n}{a} \right)^{|\alpha|},$$

where $M = \sup_{x \in B_a(\zeta)} |u(x)|$ and $\gamma_n = 2n\omega_{n-1}/((n-1)\omega_n)$.

Hint: Use the formula of Theorem 7.2, successively to the k th derivatives in balls with radius $a(|\alpha| - k)/m$, $k = o, 1, \ldots, m-1$.

16. Use the result of the previous exercise to show that $u \in C^2(\Omega)$ satisfying $\triangle u = 0$ in Ω is real analytic in Ω.

Hint: Use Stirling's formula

$$n! = n^n \mathrm{e}^{-n} \left(\sqrt{2\pi n} + O\left(\frac{1}{\sqrt{n}} \right) \right)$$

as $n \to \infty$, to show that u is in the class $C_{K,r}(\zeta)$, where $K = cM$ and $r = a/(\mathrm{e}\gamma_n)$. The constant c is the constant in the estimate $n^n \le c\mathrm{e}^n n!$ which follows from Stirling's formula. See Section 3.5 for the definition of a real analytic function.

17. Assume Ω is connected and $u \in C^2(\Omega)$ is a solution of $\triangle u = 0$ in Ω. Prove that $u \equiv 0$ in Ω if $D^\alpha u(\zeta) = 0$ for all α, for a point $\zeta \in \Omega$. In particular, $u \equiv 0$ in Ω if $u \equiv 0$ in an open subset of Ω.

18. Let $\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \ x_3 > 0\}$, which is a half-space of $\mathbb{R}^3$. Show that

$$G(x,y) = \frac{1}{4\pi|x-y|} - \frac{1}{4\pi|x-\overline{y}|},$$

where $\overline{y} = (y_1, y_2, -y_3)$, is the Green function to Ω.

19. Let $\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \ x_1^2 + x_2^2 + x_3^2 < R^2, \ x_3 > 0\}$, which is half of a ball in $\mathbb{R}^3$. Show that

$$\begin{aligned} G(x,y) &= \frac{1}{4\pi|x-y|} - \frac{R}{4\pi|y||x-y^\star|} \\ &\quad - \frac{1}{4\pi|x-\overline{y}|} + \frac{R}{4\pi|y||x-\overline{y}^\star|}, \end{aligned}$$

where $\overline{y} = (y_1, y_2, -y_3)$, $y^\star = R^2 y/(|y|^2)$ and $\overline{y}^\star = R^2\overline{y}/(|y|^2)$, is the Green function to Ω.

20. Let $\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \ x_2 > 0, \ x_3 > 0\}$, which is a wedge in $\mathbb{R}^3$. Show that
$$\begin{aligned} G(x,y) &= \frac{1}{4\pi|x-y|} - \frac{1}{4\pi|x-\overline{y}|} \\ &\quad -\frac{1}{4\pi|x-y'|} + \frac{1}{4\pi|x-\overline{y}'|}, \end{aligned}$$
where $\overline{y} = (y_1, y_2, -y_3)$, $y' = (y_1, -y_2, y_3)$ and $\overline{y}' = (y_1, -y_2, -y_3)$, is the Green function to Ω.

21. Find Green's function for the exterior of a disk, i. e., of the domain $\Omega = \{x \in \mathbb{R}^2 : \ |x| > R\}$.

22. Find Green's function for the angle domain $\Omega = \{z \in \mathbb{C} : \ 0 < \arg z < \alpha\pi\}$, $0 < \alpha < \pi$.

23. Find Green's function for the slit domain $\Omega = \{z \in \mathbb{C} : \ 0 < \arg z < 2\pi\}$.

24. Let for a sufficiently regular domain $\Omega \in \mathbb{R}^n$, a ball or a quadrangle for example,
$$F(x) = \int_\Omega K(x,y)\ dy,$$
where $K(x,y)$ is continuous in $\overline{\Omega} \times \overline{\Omega}$ where $x \neq y$, and which satisfies
$$|K(x,y)| \leq \frac{c}{|x-y|^\alpha}$$
with a constants c and α, $\alpha < n$.

    Show that $F(x)$ is continuous on $\overline{\Omega}$.

25. Prove (i) of the lemma of Section 7.5.

    Hint: Consider the case $n \geq 3$. Fix a function $\eta \in C^1(\mathbb{R})$ satisfying $0 \leq \eta \leq 1$, $0 \leq \eta' \leq 2$, $\eta(t) = 0$ for $t \leq 1$, $\eta(t) = 1$ for $t \geq 2$ and consider for $\epsilon > 0$ the regularized integral
$$V_\epsilon(x) := \int_\Omega f(y)\eta_\epsilon \frac{dy}{|x-y|^{n-2}},$$
where $\eta_\epsilon = \eta(|x-y|/\epsilon)$. Show that V_ϵ converges uniformly to V on compact subsets of $\mathbb{R}^n$ as $\epsilon \to 0$, and that $\partial V_\epsilon(x)/\partial x_i$ converges uniformly on compact subsets of $\mathbb{R}^n$ to
$$\int_\Omega f(y) \frac{\partial}{\partial x_i}\left(\frac{1}{|x-y|^{n-2}}\right)\ dy$$

as $\epsilon \to 0$.

26. Consider the inhomogeneous Dirichlet problem $-\triangle u = f$ in Ω, $u = \phi$ on $\partial\Omega$. Transform this problem into a Dirichlet problem for the Laplace equation.

    Hint: Set $u = w + v$, where $w(x) := \int_\Omega \ s(|x-y|)f(y)\ dy$.

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